Average entropy of a subsystem over a global unitary orbit of a mixed bipartite state

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Abstract

We investigate the average entropy of a subsystem within a global unitary orbit of a given mixed bipartite state in the finite-dimensional space. Without working out the closed-form expression of such average entropy for the mixed state case, we provide an analytical lower bound for this average entropy. In deriving this analytical lower bound, we get some useful by-products of independent interest. We also apply these results to estimate average correlation along a global unitary orbit of a given mixed bipartite state. When the notion of von Neumann entropy is replaced by linear entropy, the similar problem can be considered also, and moreover the exact average linear entropy formula is derived for a subsystem over a global unitary orbit of a mixed bipartite state.

Keywords: quantum state; unitary orbit; average entropy; average correlation; Page's formula

1 Introduction

In 1978, Lubkin [15] proposed a method of approximating the average entropy for a subsystem of a finite-dimensional quantum system in a global pure bipartite state by expanding the entropy as a series in terms of the average traces of powers of the system's reduced density operator, but the convergence of this series was never established. However, the author of recent paper [6] found an exact closed form expression for the average traces, in which Dyer gave a characterization of the convergence of the series.

In fact, Page conjectured in [18] that if a quantum system of Hilbert space dimension mn is in a random pure bipartite state, the average entropy of a subsystem of dimension $m \leqslant n$ should be given by the simple

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and elegant formula

$$S_{m,n} = H_{mn} - H_n - \frac{m-1}{2n},\tag{1.1}$$

where $H_k := \sum_{j=1}^{k} \frac{1}{j}$ is the k-th harmonic number. The average entropy $S_{m,n}$ in (1.1) is also served as a way of understanding the information in black hole radiation. This formula was first proved by Foong and Kanno [7] by using Fourier transform, and next by Sánchez-Ruiz [20] and by Sen [19] by using random matrix theory connected with generalized Laguerre polynomials. Some years later, Lachal [14] used a probabilistic approach to give a re-derivation of Page's formula. Recently, Zhang [23] has shown that if a quantum system of Hilbert space dimension mn is in a random pure bipartite state, the average diagonal entropy of a subsystem of dimension $m \leqslant n$ should be given by the simple and elegant formula

$$S_{m,n}^D = H_{mn} - H_n.\tag{1.2}$$

Based on the above mentioned formulas, i.e. (1.1) and (1.2), he derives quickly that the average coherence of a subsystem of dimension $m \leqslant n$ is given by $S_{m,n}^D - S_{m,n} = \frac{m-1}{2n}$ (see also [26] for another approach).

We know that a random pure state can be generated by a unitary operator chosen uniformly according to Haar measure μ. Thus the above problem can be equivalently described as follows: Consider a complex quantum system AB which consists of two subsystems A and B. For a given pure state $\rho_{AB} = |\psi_{AB}\rangle\langle\psi_{AB}|$,

$$\int_{\mathsf{U}(d_A d_B)} \mathsf{S}(\mathrm{Tr}_B(U\rho_{AB}U^\dagger))\mathrm{d}\mu(U) = H_{d_A d_B} - H_{d_B} - \frac{d_A - 1}{2d_B} \quad (d_A \leqslant d_B),$$

where $\mathsf{S}(\rho) := -\mathrm{Tr}\,(\rho\ln\rho)$ is the von Neumann entropy. Along this line, in a very recent paper [3], Christandl *et al* computed exactly the eigenvalue distributions of reduced density matrices of multipartite pure state by employing symplectic geometric method. Here we ask: can one have an analogous formula for a given mixed bipartite state ρ_{AB}? This question has been paid no or little attention to the best of our knowledge. In this note, we will make an attempt to determine the average entropy of a subsystem along a global unitary orbit of a given mixed bipartite state.

In fact, recently, many researchers studied various problems along a unitary orbit of a quantum state. For example, the total correlation attained between the subsystems of a bipartite quantum system is constrained if the bipartite system undergoes global unitary evolutions. The authors of Ref [11, 12] investigated related problems motivated by some considerations in the field of quantum thermodynamics. They have not only obtained the value of the maximal quantum mutual information (QMI), but also the maximum QMI state in the balanced bipartite quantum systems. Unlike the maximum QMI case, finding the minimum QMI state on the unitary orbit is more difficult than finding the maximum QMI state in general. Luckily, they completely solved this minimum QMI state for two-qubit case. Besides, Zhang and Fei investigated relative entropy and fidelity between two unitary orbits of two states, respectively [24]. They have obtained a lot of compact expressions for some extremal values under consideration. We have already known that any bipartite quantum state can be diagonalized under the global unitary conjugation but cannot be achieved in general under the local unitary conjugations. Because of this, Zhang *et al* considered the fidelity between one bipartite quantum state and another one undergoing local unitary dynamics [25]. The problems are related to the geometric measure of entanglement and the distillability problem.

Besides the above mentioned works, there is also one, where Oszmaniec and Kuś [17] estimated the fraction of noncorrelated states within a unitary orbit of a given mixed bipartite state. Based on the obtained result, they have proven that the fraction of noncorrelated states tends to zero exponentially fast with the dimension of the relevant Hilbert space whenever the purity exceeding some critical value. Consequently, a state within a global unitary orbit of a given bipartite state is asymptotically a correlated one. Motivated by this and Page's average entropy formula, in the present paper, we will consider the calculation of average entropy along a global unitary orbit of a given mixed bipartite state. On the one hand, our attempt made here can be seen as a generalization of Page's formula. On the other hand, the obtained result can be viewed as an estimate for the entanglement within a global unitary orbit of a given mixed bipartite state since a random state within a global unitary orbit of a given bipartite state is asymptotically a correlated one. In deriving our main results, we get a lot of by-products which is of independent interest.

The paper is organized as follows. In Sect. 2, we introduce the notion of unitary orbit for quantum states. In calculating the average entropy of a subsystem along a global unitary orbit of a given mixed bipartite state, we need to calculate some integral (Lemma 2.1), which is very important in this paper, over unitary groups. Then, some implications are discussed in Sect. 3. Specifically, we have obtained the following results: (i) For a bipartite quantum system, maximally mixed state can be represented by a uniform probability mixing of tensor products of two marginal states at each point within the global unitary orbit of any given mixed bipartite state; (ii) We use the relative entropy and fidelity as a figure of merit for correlation, and estimate the average correlation along a global unitary orbit from above and below since analytical calculation seems unavailable; (iii) A detailed research is given to the sum of average entropies of two subsystems within a global unitary orbit. We summarize the main contents of this paper in Sect. 4. Finally, we present the detailed proofs of Lemma 2.1, Theorem 2.2, Proposition 3.2, and Theorem 3.4 in Appendix.

2   Average entropy of a subsystem along a global unitary orbit

Define *unitary orbit* $\mathcal{U}_\rho$ of a given quantum state ρ on a d-dimensional Hilbert space $\mathcal{H}_d$ as follows:

$$\mathcal{U}_\rho := \left\{ U\rho U^\dagger : U \in \mathrm{U}\left(\mathcal{H}_d\right) \right\}.$$

Choose any $\rho'_{AB} \in \mathcal{U}_{\rho_{AB}}$ for a given bipartite state ρ_{AB} and $\rho'_A = \mathrm{Tr}_B(\rho'_{AB})$. Let $\Phi = \mathrm{Tr}_B$, then $\Phi^* = \otimes \mathbb{1}_B$. Assume that $\Gamma := \Phi^*\Phi$. That is $\Gamma(X) = \mathrm{Tr}_B(X) \otimes \mathbb{1}_B$.

In what follows, we compute the average entropy along the unitary orbit $\mathcal{U}_{\rho_{AB}}$ of ρ_{AB}. By the definition of entropy, $S(\rho'_A) = -\mathrm{Tr}\left(\rho'_A \ln \rho'_A\right)$, which can be rewritten as:

$$S(\rho'_A) = -\mathrm{Tr}\left(\rho_{AB} \ln\left[U^\dagger \Gamma(U\rho_{AB}U^\dagger)U\right]\right).$$

Note that $-\ln x = \sum_{n=1}^\infty \frac{1}{n}(1-x)^n$, and it follows that

$$
\begin{aligned}
-\ln\left[U^\dagger \Gamma(U\rho_{AB}U^\dagger)U\right] &= \sum_{n=1}^\infty \frac{1}{n}\left(\mathbb{1}_A \otimes \mathbb{1}_B - \left[U^\dagger \Gamma(U\rho_{AB}U^\dagger)U\right]\right)^n \\
&= \sum_{n=1}^\infty \frac{1}{n} U^\dagger \left[\Gamma(UTU^\dagger)\right]^n U,
\end{aligned}
$$

3

where $T := \mathbb{1}_A \otimes \mathbb{1}_B / d_B - \rho_{AB}$ with $\text{Tr}\,(T) = d_A - 1$. Thus

$$\int_{\text{U}(d)} S(\rho'_A) d\mu(U) = \sum_{n=1}^{\infty} \frac{1}{n} \text{Tr}\left(\rho \int U^\dagger \left[\Gamma(UTU^\dagger)\right]^n U d\mu(U)\right),$$

where $d = d_A d_B$. Denote

$$a_n := \text{Tr}\left(\rho \int U^\dagger \left[\Gamma(UTU^\dagger)\right]^n U d\mu(U)\right).$$

Hence

$$\ln(d_A) \geqslant \int S(\rho'_A) d\mu(U) = \sum_{n=1}^{\infty} \frac{a_n}{n}. \tag{2.1}$$

We see from this formula

$$\lim_{n \to \infty} a_n = 0. \tag{2.2}$$

Theoretically, although the specific formula for a_n can be obtained [4, 5, 22], however the specific form of a_n will be rather complicated. We can truncate this series (via keeping the first n terms) to obtain an approximation about the average entropy. That is,

$$\ln(d_A) \geqslant \int S(\rho'_A) d\mu(U) \geqslant a_1 + \frac{1}{2} a_2 + \cdots + \frac{1}{n} a_n. \tag{2.3}$$

The more larger n is, the more tighter lower bound is. But however, the computing about a_n is becoming very complicated when $n \geqslant 3$. As an illustration, we truncate first two terms as an estimate below. More challenging task is to determine the exact value of the above series.

(i). For $n = 1$, we have [22]

$$\int U^\dagger \Gamma(UTU^\dagger) U d\mu(U) = \frac{d\,\text{Tr}\,(\Gamma(\mathbb{1}_d)) - \text{Tr}\,(\Gamma)}{d(d^2 - 1)} \text{Tr}\,(T) \mathbb{1}_d + \frac{d\,\text{Tr}\,(\Gamma) - \text{Tr}\,(\Gamma(\mathbb{1}_d))}{d(d^2 - 1)} T.$$

Since $\Gamma = \Phi^* \Phi$, choose any orthonormal basis $\{|\varphi_j\rangle : j = 1, \ldots, d_B\}$ for subsystem space of B, it follows that

$$\Gamma(Z) = \sum_{i,j=1}^{d_B} M_{ij} Z M_{ij}^\dagger,$$

where $M_{ij} = \mathbb{1}_A \otimes |\varphi_i\rangle\langle\varphi_j| = M_{ji}^\dagger$, which implies that $\text{Tr}\,(\Gamma(\mathbb{1}_d)) = d_A d_B^2$, $\text{Tr}\,(\Gamma) = d_A^2 d_B$. Thus

$$\int U^\dagger \Gamma(UTU^\dagger) U d\mu(U) = \frac{d_A - 1}{d^2 - 1} \left[(1 + dd_B)\mathbb{1}_A \otimes \mathbb{1}_B - (d + d_B)\rho_{AB}\right]. \tag{2.4}$$

Therefore,

$$a_1 = \frac{d_A - 1}{d^2 - 1} \left[(1 + dd_B) - (d + d_B) \text{Tr}\left(\rho_{AB}^2\right)\right]. \tag{2.5}$$

(ii). For $n = 2$,

$$\int U^\dagger \left[\Gamma(UTU^\dagger)\right]^2 U d\mu(U) = \sum_{i,j,k,l=1}^{d_B} \int U^\dagger M_{ij} UTU^\dagger (M_{ji} M_{kl}) UTU^\dagger M_{lk} U d\mu(U)$$

$$= \sum_{i,j,l=1}^{d_B} \int U^\dagger M_{ij} UTU^\dagger M_{jl} UTU^\dagger M_{li} U d\mu(U).$$

In order to calculate above integral, we will need the following result, described as follows:

4

Lemma 2.1. *It holds that*

$$\int UAU^\dagger BUXU^\dagger CUDU^\dagger \mathrm{d}\mu(U) = \mu_1 \cdot \mathbb{1}_d + \mu_2 \cdot BC + \mu_3 \cdot CB + \mu_4 \cdot B + \mu_5 \cdot C, \tag{2.6}$$

where the coefficients $\mu_j (j = 1, \ldots, 5)$ can be found in the Appendix A.

Proof. See **Appendix A**. $\qquad\square$

This leads to the following result:

Theorem 2.2. *The average von Neumann entropy of a subsystem within the global unitary orbit of a generic mixed bipartite state ρ_{AB}, a density matrix on $\mathcal{H}_d \equiv \mathcal{H}_{d_A} \otimes \mathcal{H}_{d_B}$, is bounded from below by a quantity:*

$$\int S(\rho_A')\mathrm{d}\mu(U) \geqslant a_1 + \cdots + \frac{a_n}{n} \quad (\forall n \in \mathbb{N}), \tag{2.7}$$

where

$$a_n = \mathrm{Tr}\left(\rho \int U^\dagger \left[\Gamma(UTU^\dagger) \right]^n U \mathrm{d}\mu(U) \right) \geqslant a_1^n.$$

In particular, the first two terms can be given specifically:

$$a_1 = \frac{d_A - 1}{d^2 - 1} \left[(1 + dd_B) - (d + d_B) \, \mathrm{Tr}\left(\rho_{AB}^2 \right) \right], \tag{2.8}$$

$$a_2 = \left(f + g\frac{1}{d_B} + h\frac{1}{d_B^2} \right) - \left(g + h\frac{2}{d_B} \right) \mathrm{Tr}\left(\rho_{AB}^2 \right) + h \, \mathrm{Tr}\left(\rho_{AB}^3 \right), \tag{2.9}$$

where f, g, h are given below:

$$f = \frac{d_A(d_A^2 - 1)(d_B^2 - 1)}{(d^2 - 1)(d^2 - 4)} \left(d_A + d_B \, \mathrm{Tr}\left(\rho_{AB}^2 \right) - 2 \right) + \frac{(d^2 - 2d_A^2 - 2)(d_B^2 - 1)}{(d^2 - 1)(d^2 - 4)} (d_A - 1)^2, \tag{2.10}$$

$$g = \frac{2d(d_A - 1)(d_A^2 - 1)(d_B^2 - 1)}{(d^2 - 1)(d^2 - 4)}, \tag{2.11}$$

$$h = \frac{(d_A^2 - 1)(d_A^2 - 4)d_B^2}{(d^2 - 1)(d^2 - 4)}. \tag{2.12}$$

Moreover,

$$\int S(\rho_A')\mathrm{d}\mu(U) \geqslant -\ln(1 - a_1). \tag{2.13}$$

Proof. See **Appendix B**. $\qquad\square$

Remark 2.3. In particular, for $d_A = d_B = 2$, then $d = 4$, we have

$$a_1 = \frac{3}{5} - \frac{2}{5} \, \mathrm{Tr}\left(\rho_{AB}^2 \right), \quad a_2 = \frac{3}{10} - \frac{1}{5} \, \mathrm{Tr}\left(\rho_{AB}^2 \right). \tag{2.14}$$

This implies that

$$\ln 2 \geqslant \int S(\rho_A')\mathrm{d}\mu(U) \geqslant \max\left\{ \frac{9}{10} - \frac{3}{5} \, \mathrm{Tr}\left(\rho_{AB}^2 \right), \ln 5 - \ln\left(2 + 2\,\mathrm{Tr}\left(\rho_{AB}^2 \right) \right) \right\}. \tag{2.15}$$

It is left *open* that the explicit computing of the following integral for a given mixed bipartite state ρ_{AB}:

$$\int \ln\left(U^\dagger \Gamma(U\rho_{AB}U^\dagger)U \right) \mathrm{d}\mu(U). \tag{2.16}$$

Remark 2.4. We have already known that for any super-operator Ξ over $\mathcal{H}_d$ [22],

$$\int_{U(d)} d\mu(U)U^\dagger\Xi(UXU^\dagger)U = \frac{d\operatorname{Tr}(\Xi(\mathbb{1}_d)) - \operatorname{Tr}(\Xi)}{d(d^2-1)}\operatorname{Tr}(X)\mathbb{1}_d + \frac{d\operatorname{Tr}(\Xi) - \operatorname{Tr}(\Xi(\mathbb{1}_d))}{d(d^2-1)}X. \tag{2.17}$$

Now let $d = d_A d_B$ and $\mathcal{H}_d = \mathcal{H}_A \otimes \mathcal{H}_B$ with $\dim(\mathcal{H}_A) = d_A$ and $\dim(\mathcal{H}_B) = d_B$. Assume that $X = \rho_{AB}$, a density matrix on $\mathcal{H}_A \otimes \mathcal{H}_B$. Fixing an orthonormal basis $\{|\psi_{B,j}\rangle : j = 1, \ldots, d_B\}$ for $\mathcal{H}_B$. Suppose that $\Gamma(X) = \operatorname{Tr}_B(X) \otimes \mathbb{1}_B$. Then it can be rewritten as:

$$\Gamma(X) = \sum_{i,j=1}^{d_B}(\mathbb{1}_A \otimes |\psi_{B,i}\rangle\langle\psi_{B,j}|)X(\mathbb{1}_A \otimes |\psi_{B,j}\rangle\langle\psi_{B,i}|).$$

Clearly $\Gamma(\mathbb{1}_A \otimes \mathbb{1}_B) = d_B\mathbb{1}_A \otimes \mathbb{1}_B$, implying that

$$\operatorname{Tr}(\Gamma(\mathbb{1}_A \otimes \mathbb{1}_B)) = d_A d_B^2 \quad \text{and} \quad \operatorname{Tr}(\Gamma) = \sum_{i,j=1}^{d_B}(d_A\delta_{ij})^2 = d_A^2 d_B.$$

From the above discussion, we see that

$$\int_{U(d)} d\mu(U)U^\dagger\Gamma(U\rho_{AB}U^\dagger)U = \frac{dd_B - d_A}{d^2 - 1}\mathbb{1}_A \otimes \mathbb{1}_B + \frac{dd_A - d_B}{d^2 - 1}\rho_{AB}. \tag{2.18}$$

Denote $\rho'_{AB} = U\rho_{AB}U^\dagger$ and $\rho'_A = \operatorname{Tr}_B(\rho'_{AB})$. Then

$$\begin{aligned}
\operatorname{Tr}\left((\rho'_A)^2\right) &= \operatorname{Tr}\left((\rho'_A \otimes \mathbb{1}_B)\rho'_{AB}\right) = \operatorname{Tr}\left(\Gamma(\rho'_{AB})\rho'_{AB}\right) \\
&= \operatorname{Tr}\left(U^\dagger\Gamma(U\rho_{AB}U^\dagger)U\rho_{AB}\right).
\end{aligned}$$

Therefore

$$\left\langle\operatorname{Tr}\left((\rho'_A)^2\right)\right\rangle := \int d\mu(U)\operatorname{Tr}\left((\rho'_A)^2\right) = \operatorname{Tr}\left(\int d\mu(U)U^\dagger\Gamma(U\rho_{AB}U^\dagger)U\rho_{AB}\right).$$

That is,

$$\left\langle\operatorname{Tr}\left((\rho'_A)^2\right)\right\rangle = \frac{dd_B - d_A}{d^2 - 1} + \frac{dd_A - d_B}{d^2 - 1}\operatorname{Tr}\left(\rho_{AB}^2\right). \tag{2.19}$$

If we use the notion of linear entropy $S_L(\rho) := 1 - \operatorname{Tr}(\rho^2)$, then we have seen from Eq. (2.19) that the average *linear entropy* is given precisely by

$$\int S_L(\rho'_A)d\mu(U) = \frac{(d_A - 1)(d_B - 1)}{d + 1} + \frac{dd_A - d_B}{d^2 - 1}S_L(\rho_{AB}). \tag{2.20}$$

We see from this that

$$\int S_L(\rho'_B)d\mu(U) = \frac{(d_A - 1)(d_B - 1)}{d + 1} + \frac{dd_B - d_A}{d^2 - 1}S_L(\rho_{AB}). \tag{2.21}$$

Moreover for a mixed state ρ_{AB}, $\int S_L(\rho'_A)d\mu(U) = \int S_L(\rho'_B)d\mu(U)$ if and only if $d_A = d_B$. We also see that

$$\int(S_L(\rho'_A) + S_L(\rho'_B))d\mu(U) = 2\frac{(d_A - 1)(d_B - 1)}{d + 1} + \frac{d_A + d_B}{d + 1}S_L(\rho_{AB}). \tag{2.22}$$

In particular, if $d_A = d_B = 2$, then

$$\int S_L(\rho'_A)d\mu(U) = \int S_L(\rho'_B)d\mu(U) = \frac{1}{5} + \frac{2}{5}S_L(\rho_{AB}). \tag{2.23}$$

3 Some implications

Some consequences of Lemma 2.1 can be presented as follows: Let $\Phi = \mathrm{Tr}_B$ and $\Psi = \mathrm{Tr}_A$. Then $\Phi^* = \otimes \mathbb{1}_B$ and $\Psi^* = \mathbb{1}_A \otimes$. Thus for $\rho'_{AB} = U\rho_{AB}U^\dagger$,

$$\rho'_A \otimes \rho'_B = \Phi^*\Phi(U\rho_{AB}U^\dagger)\Psi^*\Psi(U\rho_{AB}U^\dagger) \tag{3.1}$$

$$= \sum_{i,j=1}^{d_B} \sum_{\mu,\nu=1}^{d_A} (\mathbb{1}_A \otimes |i\rangle\langle j|)U\rho_{AB}U^\dagger(|\mu\rangle\langle\nu| \otimes |j\rangle\langle i|)U\rho_{AB}U^\dagger(|\nu\rangle\langle\mu| \otimes \mathbb{1}_B), \tag{3.2}$$

and it follows that

$$\int(\rho'_A \otimes \rho'_B)\mathrm{d}\mu(U) = \sum_{i,j=1}^{d_B} \sum_{\mu,\nu=1}^{d_A} (\mathbb{1}_A \otimes |i\rangle\langle j|)\left(\int U\rho_{AB}U^\dagger(|\mu\rangle\langle\nu| \otimes |j\rangle\langle i|)U\rho_{AB}U^\dagger\mathrm{d}U\right)(|\nu\rangle\langle\mu| \otimes \mathbb{1}_B),$$

leading to the following identity

$$\int(\rho'_A \otimes \rho'_B)\mathrm{d}\mu(U) = \frac{\mathbb{1}_A}{d_A} \otimes \frac{\mathbb{1}_B}{d_B}. \tag{3.3}$$

This amounts to say that

Proposition 3.1. *For a bipartite quantum system, maximally mixed state can be represented by a uniform probability mixing of tensor products of two marginal states at each point within the global unitary orbit of any mixed bipartite state.*

Besides, we also get that

$$\int U^\dagger(\rho'_A \otimes \rho'_B)U\mathrm{d}\mu(U)$$

$$= \sum_{i,j=1}^{d_B} \sum_{\mu,\nu=1}^{d_A} \int U^\dagger(\mathbb{1}_A \otimes |i\rangle\langle j|)U\rho_{AB}U^\dagger(|\mu\rangle\langle\nu| \otimes |j\rangle\langle i|)U\rho_{AB}U^\dagger(|\nu\rangle\langle\mu| \otimes \mathbb{1}_B)U\mathrm{d}\mu(U),$$

leading to the following identity

$$\int U^\dagger(\rho'_A \otimes \rho'_B)U\mathrm{d}\mu(U) = c_0 \cdot \mathbb{1}_d + c_1 \cdot \rho_{AB} + c_2 \cdot \rho^2_{AB}, \tag{3.4}$$

where

$$c_0 = \frac{(d_A^2 - 1)(d_B^2 - 1)}{(d^2 - 1)(d^2 - 4)}\left(d - 2\mathrm{Tr}\left(\rho^2_{AB}\right)\right), \tag{3.5}$$

$$c_1 = \frac{d^2(d_A^2 + d_B^2 - 6) + 4}{(d^2 - 1)(d^2 - 4)}, \tag{3.6}$$

$$c_2 = \frac{2d(d_A^2 - 1)(d_B^2 - 1)}{(d^2 - 1)(d^2 - 4)}. \tag{3.7}$$

The total correlation attained between the subsystems of a bipartite quantum system is constrained if the bipartite system undergoes global unitary evolutions. The authors of Ref [11, 12] investigated related problems from the field of quantum thermodynamics. Since knowing the maximal possible variation in

correlations is useful, it raises the optimization problem, where a search of the maximal and minimal correlated states on a unitary orbit is needed. This is completely solved for two-qubit systems.

In the following, we make an attempt to calculate the average quantum mutual information (QMI) along a unitary orbit of a generic mixed bipartite state. We provide some analytical upper and/or lower bounds for the average QMI and quantum fidelity along a unitary orbit, although we cannot obtain precise formulas for them.

Proposition 3.2. *A lower bound for the maximal correlation (defined by relative entropy) within the global unitary orbit of a mixed bipartite state ρ_{AB} is given as follows*

$$\int I(A:B)_{\rho'}d\mu(U) \geqslant S\left(\rho_{AB}||c_0 \cdot \mathbb{1}_d + c_1 \cdot \rho_{AB} + c_2 \cdot \rho_{AB}^2\right), \tag{3.8}$$

where $I(A:B)_\rho := S(\rho_{AB}||\rho_A \otimes \rho_B)$, where $S(\rho||\sigma) := \text{Tr}\left(\rho(\ln\rho - \ln\sigma)\right)$ is the relative entropy. Similarly, the lower and upper bounds for the average fidelity within the global unitary orbit of a mixed bipartite state ρ_{AB} are given as follows

$$c_0 + c_1 \text{Tr}\left(\rho_{AB}^2\right) + c_2 \text{Tr}\left(\rho_{AB}^3\right) \leqslant \int F(\rho'_{AB}, \rho'_A \otimes \rho'_B)d\mu(U) \leqslant F(\rho_{AB}, c_0 \cdot \mathbb{1}_d + c_1 \cdot \rho_{AB} + c_2 \cdot \rho_{AB}^2). \tag{3.9}$$

where $F(\rho, \sigma) := \text{Tr}\left(\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right)$ is the fidelity between two states ρ and σ.

Proof. See **Appendix C**. $\square$

Remark 3.3. Recently, in [9] the authors introduced a general measure of correlations for two-qubit states based on the classical mutual information between local observables. They focus on (classical) correlations between sets of local observables instead of the quantum vs classical distinction. In this perspective, quantum states can be characterized as a whole by the average amount of (classical) correlations between all pairs of local observables. Under some restrictions, the authors calculated the average mutual information, whose value depends on the state purity, and the symmetry of the correlations distribution.

For the two-qubit case, we apply the spectral decomposition $\rho_{AB} = U\Lambda U^\dagger$ with $\Lambda = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, where

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)^\top \in \Delta_3 = \left\{(p_1, p_2, p_3, p_4)^\top \in \mathbb{R}^4 : p_j \geqslant 0(\forall j = 1, 2, 3, 4), \sum_{j=1}^{4} p_j = 1\right\}.$$

Due to the invariance under unitary conjugation, we have

$$F(\Lambda, c_0 + c_1\Lambda + c_2\Lambda^2) = \text{Tr}\left(\sqrt{\sqrt{\Lambda}(c_0 + c_1\Lambda + c_2\Lambda^2)\sqrt{\Lambda}}\right) = \sum_{j=1}^{4}(c_0\lambda_j + c_1\lambda_j^2 + c_2\lambda_j^3)^{\frac{1}{2}}.$$

The lower bound in (3.9) becomes

$$c_0 + c_1 \text{Tr}\left(\Lambda^2\right) + c_2 \text{Tr}\left(\Lambda^3\right) = c_0 + \sum_{j=1}^{4}(c_1\lambda_j^2 + c_2\lambda_j^3).$$

We numerically check the bounds in (3.9) by choosing 10000 random points in the probability simplex Δ_3 and drawing the 3D scatter plot of $(\mathcal{P}(\Lambda), c_0 + c_1 \text{Tr}\left(\Lambda^2\right) + c_2 \text{Tr}\left(\Lambda^3\right), F(\Lambda, c_0 + c_1\Lambda + c_2\Lambda^2))$. For

clarity, its three-view drawings are also given by using the corresponding 2D scatter plots: $\mathcal{P}(\Lambda)$ vs. $F(\Lambda, c_0 + c_1\Lambda + c_2\Lambda^2)$ in Figure 1(b); $c_0 + c_1 \operatorname{Tr}(\Lambda^2) + c_2 \operatorname{Tr}(\Lambda^3)$ vs. $F(\Lambda, c_0 + c_1\Lambda + c_2\Lambda^2)$ in Figure 1(c); and $\mathcal{P}(\Lambda)$ vs. $c_0 + c_1 \operatorname{Tr}(\Lambda^2) + c_2 \operatorname{Tr}(\Lambda^3)$ in Figure 1(d). We also notice that the maximum of $c_0 + c_1 \operatorname{Tr}(\Lambda^2) + c_2 \operatorname{Tr}(\Lambda^3)$ is even less than the minimum of $F(\Lambda, c_0 + c_1\Lambda + c_2\Lambda^2)$.
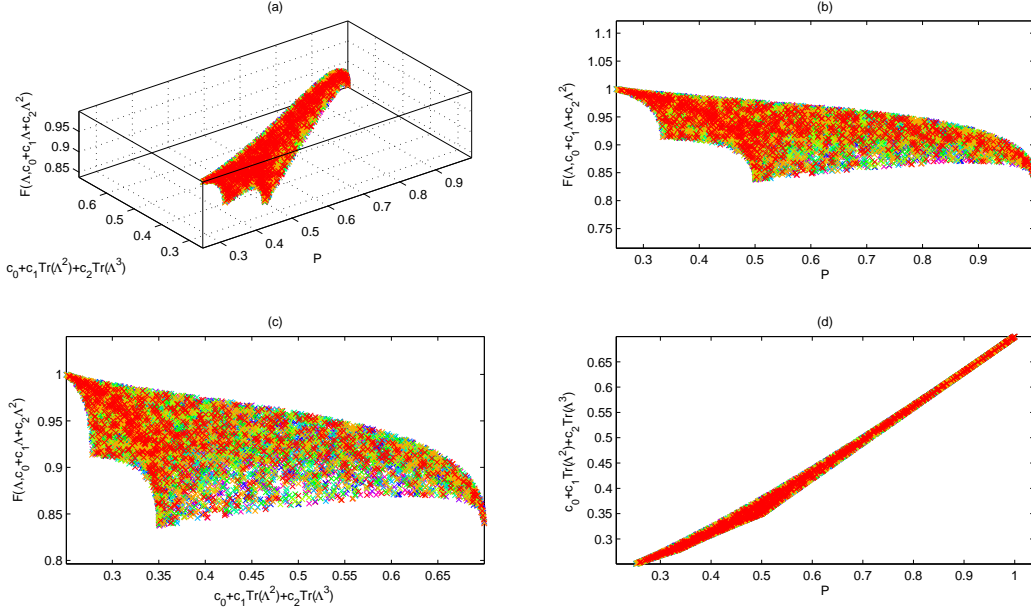

Figure 1: Two-qubit case. (a) 3D scatter plot of ($\mathcal{P}(\Lambda), c_0 + c_1 \operatorname{Tr}(\Lambda^2) + c_2 \operatorname{Tr}(\Lambda^3)$, $F(\Lambda, c_0 + c_1\Lambda + c_2\Lambda^2)$). 2D scatter plots: (b) $\mathcal{P}(\Lambda)$ vs. $F(\Lambda, c_0 + c_1\Lambda + c_2\Lambda^2)$; (c) $c_0 + c_1 \operatorname{Tr}(\Lambda^2) + c_2 \operatorname{Tr}(\Lambda^3)$ vs. $F(\Lambda, c_0 + c_1\Lambda + c_2\Lambda^2)$; (d) $\mathcal{P}(\Lambda)$ vs. $c_0 + c_1 \operatorname{Tr}(\Lambda^2) + c_2 \operatorname{Tr}(\Lambda^3)$.

In fact, not only the value of the maximal QMI is obtained, but also the maximum QMI state $\rho_{\max}$ is derived in [11, 12]. Specifically, $\rho_{AB} = U\Lambda U^\dagger$ for $d_A = d_B$, where $\Lambda = \{\lambda_j : j = 1, \ldots, d = d_A d_B\}$ is the spectrum of ρ_{AB},

$$\max_{\rho'_{AB} \in \mathcal{U}_{\rho_{AB}}} I(A:B)_{\rho'} = I(A:B)_{\rho_{\max}} = \ln(d) - S(\Lambda), \tag{3.10}$$

where

$$\rho_{\max} = \sum_{j=1}^{d} \lambda_j |\Omega_j\rangle\langle\Omega_j| \tag{3.11}$$

for any generalized Bell-state basis $\{|\Omega_j\rangle\}$. Unlike the maximum QMI case, finding $\rho_{\min}$ on the unitary orbit is more difficult than finding $\rho_{\max}$ in general. Since QMI varies on a unitary orbit completely depends on the sum of two marginal entries, we need to figure out the spectra of two reduced states of all the states in the unitary orbit $\mathcal{U}_\rho$. The literature indicates that calculating this set of compatible reduced states with a given spectrum of a global bipartite seems unforeseeable. This is well-known "quantum marginal problem", which is fully solved for two-qubit case [1] and theoretically for two-qubit case [13]. Specifically,

9

for two-qubit case, the solution of quantum marginal problem is given by the following: Mixed two-qubit state ρ_{AB} with spectrum $\Lambda = \{\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4 \geqslant 0\}$ and the marginal states ρ_A and ρ_B exist if and only if minimal eigenvalues $\lambda^A_{\min}$ and $\lambda^B_{\min}$ of the marginal states satisfy the inequalities [1]:

$$
\begin{cases}
\min\left(\lambda^A_{\min}, \lambda^B_{\min}\right) \geqslant \lambda_3 + \lambda_4, \\
\lambda^A_{\min} + \lambda^B_{\min} \geqslant \lambda_2 + \lambda_3 + 2\lambda_4, \\
\left|\lambda^A_{\min} - \lambda^B_{\min}\right| \leqslant \min\left(\lambda_1 - \lambda_3, \lambda_2 - \lambda_4\right).
\end{cases}
\tag{3.12}
$$

With the help of this result, the value of the minimal QMI on a unitary orbit is derived for two-qubit case [11, 12]:

$$
I(A : B)_{\rho_{\min}} = h(\lambda_1 + \lambda_2) + h(\lambda_1 + \lambda_3) - S(\Lambda),
\tag{3.13}
$$

where $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4 \geqslant 0$ and $h(p) := p \ln p + (1-p)\ln(1-p)$ is the binary entropy function defined for $p \in [0,1]$, moreover two-qubit state $\rho_{\min}$ is given by

$$
\rho_{\min} = \sum_{i,j=1}^{0} \lambda_{ij}|ij\rangle\langle ij|.
\tag{3.14}
$$

Here λ_{ij} is a re-indexing of $\lambda_k (\lambda_{00} = \lambda_1, \lambda_{01} = \lambda_2, \ldots)$ and $\{|i\rangle\}, \{|j\rangle\}$ are qubit basis states for subsystems A and B.

By Eq. (3.8) and the concavity of von Neumann entropy, we can draw the following conclusion:

Theorem 3.4. *For any generic mixed bipartite state ρ_{AB}, denoting*

$$
\langle S_A + S_B \rangle := \int S(\rho'_A)\mathrm{d}\mu(U) + \int S(\rho'_B)\mathrm{d}\mu(U),
$$

we have that

$$
S(\rho_{AB}) + S\left(\rho_{AB} \| c_0 \cdot \mathbb{1}_d + c_1 \cdot \rho_{AB} + c_2 \cdot \rho^2_{AB}\right) \leqslant \langle S_A + S_B \rangle \leqslant S\left(c_0 \cdot \mathbb{1}_d + c_1 \cdot \rho_{AB} + c_2 \cdot \rho^2_{AB}\right),
\tag{3.15}
$$

where $c_j (j = 0, 1, 2)$ are given by (3.5), (3.6), and (3.7), respectively. Furthermore,

$$
S\left(\rho_{AB} \| c_0 \cdot \mathbb{1}_d + c_1 \cdot \rho_{AB} + c_2 \cdot \rho^2_{AB}\right) \leqslant \int I(A : B)_{\rho'}\mathrm{d}\mu(U) \leqslant S\left(c_0 \cdot \mathbb{1}_d + c_1 \cdot \rho_{AB} + c_2 \cdot \rho^2_{AB}\right) - S(\rho_{AB}).
$$

In particular, we get a universal entropy inequality:

$$
S\left(\rho_{AB} \| c_0 \cdot \mathbb{1}_d + c_1 \cdot \rho_{AB} + c_2 \cdot \rho^2_{AB}\right) \leqslant S\left(c_0 \cdot \mathbb{1}_d + c_1 \cdot \rho_{AB} + c_2 \cdot \rho^2_{AB}\right) - S(\rho_{AB}).
\tag{3.16}
$$

Moreover, $\rho_{AB} = c_0 \cdot \mathbb{1}_d + c_1 \cdot \rho_{AB} + c_2 \cdot \rho^2_{AB}$ if and only if ρ_{AB} is maximally mixed state, i.e., $\rho_{AB} = \frac{\mathbb{1}_A}{d_A} \otimes \frac{\mathbb{1}_B}{d_B}$.

Proof. See **Appendix D**. $\qquad\square$

This result tells us whenever the state ρ_{AB} is not maximally mixed state, we can always find correlated states on a global unitary orbit of this state. Indeed, all quantum states are unitarily connected to classical states, one-way or fully classical [16].

In particular, for the balanced bipartite system, i.e., $d_A = d_B$, we have, for $d = d_A d_B$,

$$\int S(\rho'_A)\mathrm{d}\mu(U) + \int S(\rho'_B)\mathrm{d}\mu(U) \;\leqslant\; \max_U \left(S(\rho'_A) + S(\rho'_B)\right) = \max_U I(A:B)_{\rho'} + S(\rho_{AB}) = \ln(d),$$

where we used the fact that $\max_U I(A:B)_{\rho'} = \ln(d) - S(\rho_{AB})$ [11]. Apparently, this upper bound is true for $d_A \neq d_B$, and moreover it is trivially since $S(\rho'_X) \leqslant \ln(d_X)$, where $X = A, B$. Thus the following inequality is always true

$$\int S(\rho'_A)\mathrm{d}\mu(U) + \int S(\rho'_B)\mathrm{d}\mu(U) \leqslant \ln(d). \tag{3.17}$$

In view of this, we get a tighter upper bound for the sum of the average entropy of two subsystems:

$$\int S(\rho'_A)\mathrm{d}\mu(U) + \int S(\rho'_B)\mathrm{d}\mu(U) \leqslant S\left(c_0 \cdot \mathbb{1}_d + c_1 \cdot \rho_{AB} + c_2 \cdot \rho^2_{AB}\right) \leqslant \ln(d). \tag{3.18}$$

We will check the difference between the upper bound $S\left(c_0 \cdot \mathbb{1}_d + c_1 \cdot \rho_{AB} + c_2 \cdot \rho^2_{AB}\right)$ and the lower bound $S(\rho_{AB}) + S\left(\rho_{AB}||c_0 \cdot \mathbb{1}_d + c_1 \cdot \rho_{AB} + c_2 \cdot \rho^2_{AB}\right)$ since we want to identify the range of the sum of two average entropy. Let

$$\mathcal{F}(\rho_{AB}) := S\left(c_0 \cdot \mathbb{1}_d + c_1 \cdot \rho_{AB} + c_2 \cdot \rho^2_{AB}\right) - S(\rho_{AB}) - S\left(\rho_{AB}||c_0 \cdot \mathbb{1}_d + c_1 \cdot \rho_{AB} + c_2 \cdot \rho^2_{AB}\right) \tag{3.19}$$

and $\mathcal{P} = \mathrm{Tr}\left(\rho^2_{AB}\right)$, which lies in $\left[d^{-1}, 1\right]$ for $d = d_A d_B$. Apparently, $\mathcal{F} \geqslant 0$ over the whole set of states.

If $d_A = d_B = 2$, then

$$c_0 = \frac{2 - \mathcal{P}}{10}, \quad c_1 = \frac{1}{5}, \quad c_2 = \frac{2}{5}. \tag{3.20}$$

Thus in the two-qubit case, we have

$$\int U^\dagger(\rho'_A \otimes \rho'_B)U\mathrm{d}\mu(U) = \frac{2 - \mathcal{P}}{10} \cdot \mathbb{1}_4 + \frac{1}{5} \cdot \rho_{AB} + \frac{2}{5} \cdot \rho^2_{AB}. \tag{3.21}$$

Note that $S(\rho_{AB}) + S\left(\rho_{AB}||c_0 \cdot \mathbb{1}_d + c_1 \cdot \rho_{AB} + c_2 \cdot \rho^2_{AB}\right) = -\mathrm{Tr}\left(\rho_{AB}\ln\left[c_0 \cdot \mathbb{1}_d + c_1 \cdot \rho_{AB} + c_2 \cdot \rho^2_{AB}\right]\right)$. Now $\mathcal{F}$ is reduced to the following form:

$$\begin{aligned}
\mathcal{F}(\rho_{AB}) \;=\; & S\left(\frac{2 - \mathcal{P}}{10} \cdot \mathbb{1}_4 + \frac{1}{5} \cdot \rho_{AB} + \frac{2}{5} \cdot \rho^2_{AB}\right) \\
& + \mathrm{Tr}\left(\rho_{AB}\ln\left[\frac{2 - \mathcal{P}}{10} \cdot \mathbb{1}_d + \frac{1}{5} \cdot \rho_{AB} + \frac{2}{5} \cdot \rho^2_{AB}\right]\right).
\end{aligned} \tag{3.22}$$

Since $\mathcal{F}$ is invariant under unitary conjugation, it follows, via the spectral decomposition $\rho = U\Lambda U^\dagger$ of ρ_{AB}, where $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, that

$$\begin{aligned}
\mathcal{F}(\Lambda) \;=\; & S\left(\frac{2 - \mathcal{P}}{10} \cdot \mathbb{1}_4 + \frac{1}{5} \cdot \Lambda + \frac{2}{5} \cdot \Lambda^2\right) \\
& + \mathrm{Tr}\left(\Lambda\ln\left[\frac{2 - \mathcal{P}}{10} \cdot \mathbb{1}_4 + \frac{1}{5} \cdot \Lambda + \frac{2}{5} \cdot \Lambda^2\right]\right).
\end{aligned} \tag{3.23}$$

We see that $\mathcal{F}$ is a symmetric function defined over the probability simplex

$$\Delta_3 = \left\{(p_1, p_2, p_3, p_4)^\mathsf{T} \in \mathbb{R}^4 : p_j \geqslant 0(\forall j = 1, 2, 3, 4), \sum_{j=1}^4 p_j = 1\right\}.$$

Next we make numerical analysis of this function defined over the probability simplex.

$$\mathcal{F}(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \sum_{j=1}^{4} \lambda_j \ln\left(\frac{2-\mathcal{P}}{10} + \frac{1}{5}\lambda_j + \frac{2}{5}\lambda_j^2\right) \tag{3.24}$$

$$- \left(\frac{2-\mathcal{P}}{10} + \frac{1}{5}\lambda_j + \frac{2}{5}\lambda_j^2\right) \ln\left(\frac{2-\mathcal{P}}{10} + \frac{1}{5}\lambda_j + \frac{2}{5}\lambda_j^2\right), \tag{3.25}$$

where $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)^{\mathsf{T}} \in \Delta_3$ and $\mathcal{P} \equiv \mathcal{P}(\Lambda) := \mathrm{Tr}\left(\Lambda^2\right) = \sum_{j=1}^{4} \lambda_j^2$.

Choose random point Λ in the probability simplex Δ_3, and then draw the 3-dimensional figure about the 3-tuples (x, y, z) where $x = \mathcal{P}(\Lambda)$, $y = -\mathrm{Tr}\left(\Lambda \ln\left[\frac{2-\mathcal{P}(\Lambda)}{10} \cdot \mathbb{1}_4 + \frac{1}{5} \cdot \Lambda + \frac{2}{5} \cdot \Lambda^2\right]\right)$, $z = S(c_0 + c_1\Lambda + c_2\Lambda^2)$. To illustrate the relationship among $\mathcal{P}(\Lambda)$, $S(\rho_{AB}) + S\left(\rho_{AB} || c_0 \cdot \mathbb{1}_d + c_1 \cdot \rho_{AB} + c_2 \cdot \rho_{AB}^2\right)$ and $S(c_0 + c_1\Lambda + c_2\Lambda^2)$, we show the 3D scatter plot in Figure 2(a), where there are 5000 random points used in total. To make things more clear, we also demonstrate 2D scatter plots: $S(\rho_{AB}) + S\left(\rho_{AB} || c_0 \cdot \mathbb{1}_d + c_1 \cdot \rho_{AB} + c_2 \cdot \rho_{AB}^2\right)$ vs. $S(c_0 + c_1\Lambda + c_2\Lambda^2)$ in Figure 2(b), $\mathcal{P}$ vs. $S(c_0 + c_1\Lambda + c_2\Lambda^2)$ in Figure 2(c), and $\mathcal{P}$ vs. $\mathcal{F}$ in Figure 2(d). Note that for the case where $\lambda_j = \frac{1}{4}(j = 1, 2, 3, 4)$ we have $S(\Lambda) = S(c_0 + c_1\Lambda + c_2\Lambda^2) = \ln 4 = 1.39$ and $\mathcal{F} = 0$; while the case with one λ_j equaling to 1 and the other three being zeros yields $S(\Lambda) = 0$, $S(c_0 + c_1\Lambda + c_2\Lambda^2) = -\frac{3}{10}\ln\frac{1}{10} - \frac{7}{10}\ln\frac{7}{10} = 0.94$, $\mathrm{Tr}\left(\Lambda \ln(c_0 + c_1\Lambda + c_2\Lambda^2)\right) = \ln\frac{7}{10}$ and $\mathcal{F} = -\frac{3}{10}\ln\frac{1}{10} - \frac{3}{5}\ln\frac{7}{10} = 0.58$. In these plots, $\mathcal{P} \in [\frac{1}{4}, 1]$, $S(\Lambda) \in [0, \ln 4]$, $S(c_0 + c_1\Lambda + c_2\Lambda^2) \in [0.94, \ln 4]$ and $\mathcal{F} \in [0, 0.58]$.
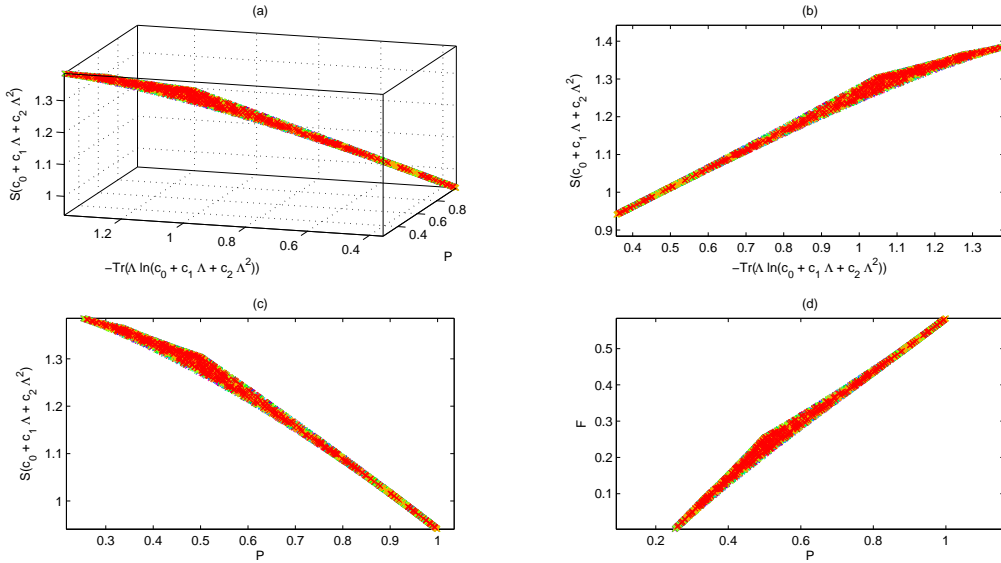

Figure 2: Two-qubit case $(d_A = d_B = 2)$. (a) 3D scatter plot of $(\mathcal{P}(\Lambda), -\mathrm{Tr}\left(\Lambda \ln(c_0 + c_1\Lambda + c_2\Lambda^2)\right), S(c_0 + c_1\Lambda + c_2\Lambda^2))$. 2D scatter plots: (b) $-\mathrm{Tr}\left(\Lambda \ln(c_0 + c_1\Lambda + c_2\Lambda^2)\right)$ vs. $S(c_0 + c_1\Lambda + c_2\Lambda^2)$; (c) $\mathcal{P}(\Lambda)$ vs. $S(c_0 + c_1\Lambda + c_2\Lambda^2)$; (d) $\mathcal{P}(\Lambda)$ vs. $\mathcal{F}(\Lambda)$.

Remark 3.5. Denote $\rho' = U\rho U^{\dagger}$ for a fixed state ρ. If U is such that $I(A : B)_{\rho'} > I(A : B)_{\rho}$, then we say that U enhances the correlation between A and B, otherwise, we say that it weakens the correlation A and B when $I(A : B)_{\rho'} < I(A : B)_{\rho}$. In particular, if $I(A : B)_{\rho'} = 0$, then we say that U decouples A

from B. Clearly there exists a state (for example completely mixed state) of which its correlations cannot be enhanced by any unitary. We see from previous discussion, finding minimal mutual information is in terms of an optimization problem that is extremely hard to handle in higher dimensions. Proposition 3.2 and Theorem 3.4 can be seen as a different strategy based on probabilistic averages. Denote

$$\Delta(\rho_{AB}) := \int I(A:B)_{\rho'} \mathrm{d}\mu(U) - I(A:B)_{\rho}.$$

If $\Delta(\rho) > 0$, then the correlations existing in ρ_{AB} can be enhanced by many unitaries (via a perspective of concentration of measure phenomenon from Measure Theory). Clearly a lot of states of product forms satisfies $\Delta(\rho) > 0$. In the opposite, $\Delta(\rho) < 0$ means that the correlations existing in ρ_{AB} can be decreased by many unitaries. Naturally, some questions arise: For a given state ρ_{AB}, one wants to know wether if its unitary orbit of ρ_{AB} contains product state(s).

Remark 3.6. A similar questions can be considered for the so-called *quantum conditional mutual information* (QCMI), defined by $I(A:B|E)_{\rho} := S(\rho_{AE}) + S(\rho_{BE}) - S(\rho_{ABE}) - S(\rho_E)$ for a tripartite ρ_{ABE} on a tensorial Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_E$. We still denote $\rho'_{ABE} = (U \otimes \mathbb{1}_E)\rho_{ABE}(U^\dagger \otimes \mathbb{1}_E)$ for any unitary U on $\mathcal{H}_A \otimes \mathcal{H}_B$. Analogously, maximal and/or minimal QCMI can be considered, that is,

$$\max_U I(A:B|E)_{\rho'} \quad \text{and} \quad \min_U I(A:B|E)_{\rho'}.$$

Clearly $\min_U I(A:B|E)_{\rho'}$ is very important since it gives a lower bound for QCMI:

$$I(A:B|E)_{\rho} \geqslant \min_U I(A:B|E)_{\rho'}.$$

We can also consider the average QCMI:

$$\int I(A:B|E)_{\rho'} \mathrm{d}\mu(U). \tag{3.26}$$

All problems mentioned above are beyond the goal of this paper. We will come back to them in the future research.

Remark 3.7. Given a quantum channel $\mathcal{E}$ (trace-preserving and completely positive linear map) which is, via Kraus representation, represented as $\mathcal{E} = \sum_j \mathrm{Ad}_{E_j}$, where $\mathrm{Ad}_{E_j}(X) := E_j X E_j^\dagger$. We can use Lemma 2.1 to get the average purity of a unitary orbit of a given state undergoing a fixed quantum channel $\mathcal{E}$. Indeed, by Choi-Jamiołkowski isomorphism [2], $J(\mathcal{E}) := (\mathcal{E} \otimes \mathbb{1})(|\mathbb{1}_d\rangle\langle\mathbb{1}_d|) = \sum_j |E_j\rangle\langle E_j|$,

$$\int \mathcal{E}(U\rho U^\dagger)^2 \mathrm{d}\mu(U) = \sum_{i,j} E_i \left(U\rho U^\dagger E_i^\dagger E_j U\rho U^\dagger \right) E_j^\dagger \tag{3.27}$$

$$= \frac{d\,\mathrm{Tr}\left(\rho^2\right) - 1}{d(d^2 - 1)} \sum_{i,j} \mathrm{Tr}\left(E_i^\dagger E_j\right) E_i E_j^\dagger + \frac{d - \mathrm{Tr}\left(\rho^2\right)}{d(d^2 - 1)} \mathcal{E}(\mathbb{1}_d)^2, \tag{3.28}$$

implying that

$$\int \mathrm{Tr}\left(\mathcal{E}(U\rho U^\dagger)^2\right) \mathrm{d}\mu(U) = \frac{d\,\mathrm{Tr}\left(\rho^2\right) - 1}{d(d^2 - 1)} \mathrm{Tr}\left(J(\mathcal{E})^2\right) + \frac{d - \mathrm{Tr}\left(\rho^2\right)}{d(d^2 - 1)} \mathrm{Tr}\left(\mathcal{E}(\mathbb{1}_d)^2\right), \tag{3.29}$$

where the lhs of the above last formula is just the average purity of a unitary orbit of a given state undergoing a fixed quantum channel $\mathcal{E}$.

4 Concluding remarks

In this paper, we investigate the average entropy of a subsystem along a global unitary orbit of a given mixed bipartite state in the finite-dimensional space. Although it is still unable to derive the closed-form compact formula for the mixed state case, compared with Page's formula, nevertheless we get an analytical lower bound for this average entropy for the mixed state case. In deriving this analytical lower bound, we obtain some useful by-products of independent interest, for instance, for a bipartite quantum system, maximally mixed state can be represented by a uniform probability mixing of tensor products of two marginal states at each point within the global unitary orbit of any given mixed bipartite state. This is amazing. In addition, from the discussion after finishing the proof of Proposition 3.2, we see that the average entropy of a subsystem is intimately related to the well-known "quantum marginal problem" or N-representability in quantum chemistry. Besides, it also connects with entanglement polytope, a notion proposed by M. Walter *et al.* in studying multipartite entanglement from single-particle information [21].

These obtained results can be applied to estimate average correlation along a global unitary orbit of a given mixed bipartite state. The corresponding numerics about these results is also provided in lower dimensional case (for instance $d_A = d_B = 2$). Except that, the results obtained in the present paper can also be used to study further the average coherence of a class of random states induced from isospectral bipartite mixed states. Indeed, recently we have already calculated exactly the average coherence for random mixed quantum states [23, 26] induced from random bipartite pure states.

Finally, we conclude this section with two open problems: (i) computing the average coherence (via relative entropy of coherence) of a subsystem of isospectral bipartite systems

$$\int \Big(S((\rho'_A)_{\text{diag}}) - S(\rho'_A) \Big) \, d\mu(U), \tag{4.1}$$

and (ii) proving the following identity for the balanced system, i.e., $d_A = d_B$,

$$\int S(\rho'_A) d\mu(U) = \int S(\rho'_B) d\mu(U). \tag{4.2}$$

If (4.2) *were* true, then we would see from Theorem 2.2 and Theorem 3.4 that

$$-\ln(1 - a_1) \leqslant \int S(\rho'_A) d\mu(U) \leqslant \frac{1}{2} S \Big(c_0 \cdot \mathbb{1}_d + c_1 \cdot \rho_{AB} + c_2 \cdot \rho_{AB}^2 \Big). \tag{4.3}$$

We hope that the present work and questions proposed can bring out more interesting and insightful perspective(s) in quantum information theory.

Appendix

A. The proof of Lemma 2.1

In this paper, we will utilize some notion of matrix integral [4, 5, 22]. The formula in Lemma 2.1 is given firstly. A detailed reasoning is presented here.

The proof of Lemma 2.1. Firstly we note that

$$
\begin{aligned}
\left\langle i_1 \left| UAU^\dagger BUXU^\dagger CUDU^\dagger \right| i_1' \right\rangle &= \sum_{j_1,j_1'} \langle i_1 |U| j_1 \rangle \left\langle j_1 \left| AU^\dagger BUXU^\dagger CUD \right| j_1' \right\rangle \left\langle j_1' \left| U^\dagger \right| i_1' \right\rangle \\
&= \sum_{i_2,j_1,j_2,i_2',j_1',j_2'} U_{i_1 j_1} \overline{U}_{i_1' j_1'} \langle j_1 |A| j_2' \rangle \left\langle j_2' \left| U^\dagger \right| i_2' \right\rangle \left\langle i_2' \left| BUXU^\dagger C \right| i_2 \right\rangle \langle i_2 |U| j_2 \rangle \langle j_2 |D| j_1' \rangle \\
&= \sum_{i_2,j_1,j_2,i_2',j_1',j_2'} U_{i_1 j_1} U_{i_2 j_2} \overline{U}_{i_1' j_1'} \overline{U}_{i_2' j_2'} \langle j_1 |A| j_2' \rangle \left\langle i_2' \left| BUXU^\dagger C \right| i_2 \right\rangle \langle j_2 |D| j_1' \rangle \\
&= \sum_{i_2,i_3,j_1,j_2,j_3,i_2',i_3',j_1',j_2',j_3'} U_{i_1 j_1} U_{i_2 j_2} \overline{U}_{i_1' j_1'} \overline{U}_{i_2' j_2'} \\
&\quad \times \langle j_1 |A| j_2' \rangle \langle i_2' |B| i_3 \rangle \langle i_3 |U| j_3 \rangle \langle j_3 |X| j_3' \rangle \left\langle j_3' \left| U^\dagger \right| i_3' \right\rangle \langle i_3' |C| i_2 \rangle \langle j_2 |D| j_1' \rangle \\
&= \sum_{i_2,i_3,j_1,j_2,j_3,i_2',i_3',j_1',j_2',j_3'} U_{i_1 j_1} U_{i_2 j_2} U_{i_3 j_3} \overline{U}_{i_1' j_1'} \overline{U}_{i_2' j_2'} \overline{U}_{i_3' j_3'} \langle j_1 |A| j_2' \rangle \langle i_2' |B| i_3 \rangle \langle j_3 |X| j_3' \rangle \langle i_3' |C| i_2 \rangle \langle j_2 |D| j_1' \rangle .
\end{aligned}
$$

Then we have:

$$
\begin{aligned}
&\left\langle i_1 \left| \int UAU^\dagger BUXU^\dagger CUDU^\dagger dU \right| i_1' \right\rangle \\
&= \sum_{i_2,i_3,j_1,j_2,j_3,i_2',i_3',j_1',j_2',j_3'} A_{j_1,j_2'} B_{i_2',i_3} X_{j_3,j_3'} C_{i_3',i_2} D_{j_2,j_1'} \left(\int U_{i_1 j_1} U_{i_2 j_2} U_{i_3 j_3} \overline{U}_{i_1' j_1'} \overline{U}_{i_2' j_2'} \overline{U}_{i_3' j_3'} dU \right) \\
&= \sum_{i_2,i_3,j_1,j_2,j_3,i_2',i_3',j_1',j_2',j_3'} A_{j_1,j_2'} B_{i_2',i_3} X_{j_3,j_3'} C_{i_3',i_2} D_{j_2,j_1'} \\
&\quad \times \left(\sum_{\pi,\sigma\in S_3} \langle i_1|i_{\pi(1)}'\rangle \langle i_2|i_{\pi(2)}'\rangle \langle i_3|i_{\pi(3)}'\rangle \langle j_1|j_{\sigma(1)}'\rangle \langle j_2|j_{\sigma(2)}'\rangle \langle j_3|j_{\sigma(3)}'\rangle \mathrm{Wg}(\sigma\pi^{-1}) \right) \\
&= \sum_{\pi,\sigma\in S_3} \mathrm{Wg}(\sigma\pi^{-1}) \\
&\quad \times \left(\sum_{i_2,i_3,j_1,j_2,j_3,i_2',i_3',j_1',j_2',j_3'} A_{j_1,j_2'} B_{i_2',i_3} X_{j_3,j_3'} C_{i_3',i_2} D_{j_2,j_1'} \langle i_1|i_{\pi(1)}'\rangle \langle i_2|i_{\pi(2)}'\rangle \langle i_3|i_{\pi(3)}'\rangle \langle j_1|j_{\sigma(1)}'\rangle \langle j_2|j_{\sigma(2)}'\rangle \langle j_3|j_{\sigma(3)}'\rangle \right) .
\end{aligned}
$$

In what follows, we compute this value step-by-step.

(1). If $(\pi,\sigma) = ((1),(1))$, then

$$
\sum_{i_2,i_3,j_1,j_2,j_3,i_2',i_3',j_1',j_2',j_3'} A_{j_1,j_2'} B_{i_2',i_3} X_{j_3,j_3'} C_{i_3',i_2} D_{j_2,j_1'} \langle i_1|i_1'\rangle \langle i_2|i_2'\rangle \langle i_3|i_3'\rangle \langle j_1|j_1'\rangle \langle j_2|j_2'\rangle \langle j_3|j_3'\rangle \tag{4.4}
$$

$$
= \mathrm{Tr}\,(AD)\,\mathrm{Tr}\,(X)\,\mathrm{Tr}\,(BC)\,\langle i_1|i_1'\rangle. \tag{4.5}
$$

(2). If $(\pi, \sigma) = ((1), (12))$, then

$$\sum_{i_2,i_3,j_1,j_2,j_3,i_2',i_3',j_1',j_2',j_3'} A_{j_1,j_2'} B_{i_2',i_3} X_{j_3,j_3'} C_{i_3',i_2} D_{j_2,j_1'} \langle i_1|i_1'\rangle \langle i_2|i_2'\rangle \langle i_3|i_3'\rangle \langle j_1|j_2'\rangle \langle j_2|j_1'\rangle \langle j_3|j_3'\rangle \tag{4.6}$$

$$= \mathrm{Tr}\,(A)\,\mathrm{Tr}\,(D)\,\mathrm{Tr}\,(X)\,\mathrm{Tr}\,(BC)\,\langle i_1|i_1'\rangle. \tag{4.7}$$

(3). If $(\pi, \sigma) = ((1), (13))$, then

$$\sum_{i_2,i_3,j_1,j_2,j_3,i_2',i_3',j_1',j_2',j_3'} A_{j_1,j_2'} B_{i_2',i_3} X_{j_3,j_3'} C_{i_3',i_2} D_{j_2,j_1'} \langle i_1|i_1'\rangle \langle i_2|i_2'\rangle \langle i_3|i_3'\rangle \langle j_1|j_3'\rangle \langle j_2|j_2'\rangle \langle j_3|j_1'\rangle \tag{4.8}$$

$$= \mathrm{Tr}\,(ADX)\,\mathrm{Tr}\,(BC)\,\langle i_1|i_1'\rangle. \tag{4.9}$$

(4). If $(\pi, \sigma) = ((1), (23))$, then

$$\sum_{i_2,i_3,j_1,j_2,j_3,i_2',i_3',j_1',j_2',j_3'} A_{j_1,j_2'} B_{i_2',i_3} X_{j_3,j_3'} C_{i_3',i_2} D_{j_2,j_1'} \langle i_1|i_1'\rangle \langle i_2|i_2'\rangle \langle i_3|i_3'\rangle \langle j_1|j_1'\rangle \langle j_2|j_3'\rangle \langle j_3|j_2'\rangle \tag{4.10}$$

$$= \mathrm{Tr}\,(DAX)\,\mathrm{Tr}\,(BC)\,\langle i_1|i_1'\rangle. \tag{4.11}$$

(5). If $(\pi, \sigma) = ((1), (123))$, then

$$\sum_{i_2,i_3,j_1,j_2,j_3,i_2',i_3',j_1',j_2',j_3'} A_{j_1,j_2'} B_{i_2',i_3} X_{j_3,j_3'} C_{i_3',i_2} D_{j_2,j_1'} \langle i_1|i_1'\rangle \langle i_2|i_2'\rangle \langle i_3|i_3'\rangle \langle j_1|j_2'\rangle \langle j_2|j_3'\rangle \langle j_3|j_1'\rangle \tag{4.12}$$

$$= \mathrm{Tr}\,(A)\,\mathrm{Tr}\,(DX)\,\mathrm{Tr}\,(BC)\,\langle i_1|i_1'\rangle. \tag{4.13}$$

(6). If $(\pi, \sigma) = ((1), (132))$, then

$$\sum_{i_2,i_3,j_1,j_2,j_3,i_2',i_3',j_1',j_2',j_3'} A_{j_1,j_2'} B_{i_2',i_3} X_{j_3,j_3'} C_{i_3',i_2} D_{j_2,j_1'} \langle i_1|i_1'\rangle \langle i_2|i_2'\rangle \langle i_3|i_3'\rangle \langle j_1|j_3'\rangle \langle j_2|j_1'\rangle \langle j_3|j_2'\rangle \tag{4.14}$$

$$= \mathrm{Tr}\,(D)\,\mathrm{Tr}\,(AX)\,\mathrm{Tr}\,(BC)\,\langle i_1|i_1'\rangle. \tag{4.15}$$

(7). If $(\pi, \sigma) = ((12), (1))$, then

$$\sum_{i_2,i_3,j_1,j_2,j_3,i_2',i_3',j_1',j_2',j_3'} A_{j_1,j_2'} B_{i_2',i_3} X_{j_3,j_3'} C_{i_3',i_2} D_{j_2,j_1'} \langle i_1|i_2'\rangle \langle i_2|i_1'\rangle \langle i_3|i_3'\rangle \langle j_1|j_1'\rangle \langle j_2|j_2'\rangle \langle j_3|j_3'\rangle \tag{4.16}$$

$$= \mathrm{Tr}\,(AD)\,\mathrm{Tr}\,(X)\,\langle i_1\,|BC|\,i_1'\rangle. \tag{4.17}$$

(8). If $(\pi, \sigma) = ((12), (12))$, then

$$\sum_{i_2,i_3,j_1,j_2,j_3,i_2',i_3',j_1',j_2',j_3'} A_{j_1,j_2'} B_{i_2',i_3} X_{j_3,j_3'} C_{i_3',i_2} D_{j_2,j_1'} \langle i_1|i_2'\rangle \langle i_2|i_1'\rangle \langle i_3|i_3'\rangle \langle j_1|j_2'\rangle \langle j_2|j_1'\rangle \langle j_3|j_3'\rangle \tag{4.18}$$

$$= \mathrm{Tr}\,(A)\,\mathrm{Tr}\,(D)\,\mathrm{Tr}\,(X)\,\langle i_1\,|BC|\,i_1'\rangle. \tag{4.19}$$

(9). If $(\pi, \sigma) = ((12), (13))$, then

$$\sum_{i_2,i_3,j_1,j_2,j_3,i_2',i_3',j_1',j_2',j_3'} A_{j_1,j_2'} B_{i_2',i_3} X_{j_3,j_3'} C_{i_3',i_2} D_{j_2,j_1'} \langle i_1|i_2'\rangle \langle i_2|i_1'\rangle \langle i_3|i_3'\rangle \langle j_1|j_3'\rangle \langle j_2|j_2'\rangle \langle j_3|j_1'\rangle \tag{4.20}$$

$$= \mathrm{Tr}\,(ADX)\,\langle i_1\,|BC|\,i_1'\rangle. \tag{4.21}$$

(10). If $(\pi, \sigma) = ((12), (23))$, then

$$\sum_{i_2,i_3,j_1,j_2,j_3,i_2',i_3',j_1',j_2',j_3'} A_{j_1,j_2'} B_{i_2',i_3} X_{j_3,j_3'} C_{i_3',i_2} D_{j_2,j_1'} \langle i_1|i_2'\rangle \langle i_2|i_1'\rangle \langle i_3|i_3'\rangle \langle j_1|j_1'\rangle \langle j_2|j_3'\rangle \langle j_3|j_2'\rangle \tag{4.22}$$

$$= \mathrm{Tr}\,(DAX)\,\langle i_1\,|BC|\,i_1'\rangle. \tag{4.23}$$

(11). If $(\pi, \sigma) = ((12), (123))$, then

$$\sum_{i_2,i_3,j_1,j_2,j_3,i'_2,i'_3,j'_1,j'_2,j'_3} A_{j_1,j'_2} B_{i'_2,i_3} X_{j_3,j'_3} C_{i'_3,i_2} D_{j_2,j'_1} \langle i_1|i'_2\rangle \langle i_2|i'_1\rangle \langle i_3|i'_3\rangle \langle j_1|j'_2\rangle \langle j_2|j'_3\rangle \langle j_3|j'_1\rangle \tag{4.24}$$

$$= \mathrm{Tr}\,(A)\,\mathrm{Tr}\,(DX)\,\langle i_1\,|BC|\,i'_1\rangle. \tag{4.25}$$

(12). If $(\pi, \sigma) = ((12), (132))$, then

$$\sum_{i_2,i_3,j_1,j_2,j_3,i'_2,i'_3,j'_1,j'_2,j'_3} A_{j_1,j'_2} B_{i'_2,i_3} X_{j_3,j'_3} C_{i'_3,i_2} D_{j_2,j'_1} \langle i_1|i'_2\rangle \langle i_2|i'_1\rangle \langle i_3|i'_3\rangle \langle j_1|j'_3\rangle \langle j_2|j'_1\rangle \langle j_3|j'_2\rangle \tag{4.26}$$

$$= \mathrm{Tr}\,(D)\,\mathrm{Tr}\,(AX)\,\langle i_1\,|BC|\,i'_1\rangle. \tag{4.27}$$

(13). If $(\pi, \sigma) = ((13), (1))$, then

$$\sum_{i_2,i_3,j_1,j_2,j_3,i'_2,i'_3,j'_1,j'_2,j'_3} A_{j_1,j'_2} B_{i'_2,i_3} X_{j_3,j'_3} C_{i'_3,i_2} D_{j_2,j'_1} \langle i_1|i'_3\rangle \langle i_2|i'_2\rangle \langle i_3|i'_1\rangle \langle j_1|j'_1\rangle \langle j_2|j'_2\rangle \langle j_3|j'_3\rangle \tag{4.28}$$

$$= \mathrm{Tr}\,(AD)\,\mathrm{Tr}\,(X)\,\langle i_1\,|CB|\,i'_1\rangle. \tag{4.29}$$

(14). If $(\pi, \sigma) = ((13), (12))$, then

$$\sum_{i_2,i_3,j_1,j_2,j_3,i'_2,i'_3,j'_1,j'_2,j'_3} A_{j_1,j'_2} B_{i'_2,i_3} X_{j_3,j'_3} C_{i'_3,i_2} D_{j_2,j'_1} \langle i_1|i'_3\rangle \langle i_2|i'_2\rangle \langle i_3|i'_1\rangle \langle j_1|j'_2\rangle \langle j_2|j'_1\rangle \langle j_3|j'_3\rangle \tag{4.30}$$

$$= \mathrm{Tr}\,(A)\,\mathrm{Tr}\,(D)\,\mathrm{Tr}\,(X)\,\langle i_1\,|CB|\,i'_1\rangle. \tag{4.31}$$

(15). If $(\pi, \sigma) = ((13), (13))$, then

$$\sum_{i_2,i_3,j_1,j_2,j_3,i'_2,i'_3,j'_1,j'_2,j'_3} A_{j_1,j'_2} B_{i'_2,i_3} X_{j_3,j'_3} C_{i'_3,i_2} D_{j_2,j'_1} \langle i_1|i'_3\rangle \langle i_2|i'_2\rangle \langle i_3|i'_1\rangle \langle j_1|j'_3\rangle \langle j_2|j'_2\rangle \langle j_3|j'_1\rangle \tag{4.32}$$

$$= \mathrm{Tr}\,(ADX)\,\langle i_1\,|CB|\,i'_1\rangle. \tag{4.33}$$

(16). If $(\pi, \sigma) = ((13), (23))$, then

$$\sum_{i_2,i_3,j_1,j_2,j_3,i'_2,i'_3,j'_1,j'_2,j'_3} A_{j_1,j'_2} B_{i'_2,i_3} X_{j_3,j'_3} C_{i'_3,i_2} D_{j_2,j'_1} \langle i_1|i'_3\rangle \langle i_2|i'_2\rangle \langle i_3|i'_1\rangle \langle j_1|j'_1\rangle \langle j_2|j'_3\rangle \langle j_3|j'_2\rangle \tag{4.34}$$

$$= \mathrm{Tr}\,(DAX)\,\langle i_1\,|CB|\,i'_1\rangle. \tag{4.35}$$

(17). If $(\pi, \sigma) = ((13), (123))$, then

$$\sum_{i_2,i_3,j_1,j_2,j_3,i'_2,i'_3,j'_1,j'_2,j'_3} A_{j_1,j'_2} B_{i'_2,i_3} X_{j_3,j'_3} C_{i'_3,i_2} D_{j_2,j'_1} \langle i_1|i'_3\rangle \langle i_2|i'_2\rangle \langle i_3|i'_1\rangle \langle j_1|j'_2\rangle \langle j_2|j'_3\rangle \langle j_3|j'_1\rangle \tag{4.36}$$

$$= \mathrm{Tr}\,(A)\,\mathrm{Tr}\,(DX)\,\langle i_1\,|CB|\,i'_1\rangle. \tag{4.37}$$

(18). If $(\pi, \sigma) = ((13), (132))$, then

$$\sum_{i_2,i_3,j_1,j_2,j_3,i'_2,i'_3,j'_1,j'_2,j'_3} A_{j_1,j'_2} B_{i'_2,i_3} X_{j_3,j'_3} C_{i'_3,i_2} D_{j_2,j'_1} \langle i_1|i'_3\rangle \langle i_2|i'_2\rangle \langle i_3|i'_1\rangle \langle j_1|j'_3\rangle \langle j_2|j'_1\rangle \langle j_3|j'_2\rangle \tag{4.38}$$

$$= \mathrm{Tr}\,(D)\,\mathrm{Tr}\,(AX)\,\langle i_1\,|CB|\,i'_1\rangle. \tag{4.39}$$

(19). If $(\pi, \sigma) = ((23), (1))$, then

$$\sum_{i_2,i_3,j_1,j_2,j_3,i'_2,i'_3,j'_1,j'_2,j'_3} A_{j_1,j'_2} B_{i'_2,i_3} X_{j_3,j'_3} C_{i'_3,i_2} D_{j_2,j'_1} \langle i_1|i'_1\rangle \langle i_2|i'_3\rangle \langle i_3|i'_2\rangle \langle j_1|j'_1\rangle \langle j_2|j'_2\rangle \langle j_3|j'_3\rangle \tag{4.40}$$

$$= \mathrm{Tr}\,(AD)\,\mathrm{Tr}\,(X)\,\mathrm{Tr}\,(B)\,\mathrm{Tr}\,(C)\,\langle i_1|i'_1\rangle. \tag{4.41}$$

(20). If $(\pi, \sigma) = ((23), (12))$, then

$$\sum_{i_2,i_3,j_1,j_2,j_3,i_2',i_3',j_1',j_2',j_3'} A_{j_1,j_2'} B_{i_2',i_3} X_{j_3,j_3'} C_{i_3',i_2} D_{j_2,j_1'} \langle i_1|i_1'\rangle \langle i_2|i_3'\rangle \langle i_3|i_2'\rangle \langle j_1|j_2'\rangle \langle j_2|j_1'\rangle \langle j_3|j_3'\rangle \tag{4.42}$$

$$= \mathrm{Tr}\,(A)\,\mathrm{Tr}\,(D)\,\mathrm{Tr}\,(X)\,\mathrm{Tr}\,(B)\,\mathrm{Tr}\,(C)\,\langle i_1|i_1'\rangle. \tag{4.43}$$

(21). If $(\pi, \sigma) = ((23), (13))$, then

$$\sum_{i_2,i_3,j_1,j_2,j_3,i_2',i_3',j_1',j_2',j_3'} A_{j_1,j_2'} B_{i_2',i_3} X_{j_3,j_3'} C_{i_3',i_2} D_{j_2,j_1'} \langle i_1|i_1'\rangle \langle i_2|i_3'\rangle \langle i_3|i_2'\rangle \langle j_1|j_3'\rangle \langle j_2|j_2'\rangle \langle j_3|j_1'\rangle \tag{4.44}$$

$$= \mathrm{Tr}\,(ADX)\,\mathrm{Tr}\,(B)\,\mathrm{Tr}\,(C)\,\langle i_1|i_1'\rangle. \tag{4.45}$$

(22). If $(\pi, \sigma) = ((23), (23))$, then

$$\sum_{i_2,i_3,j_1,j_2,j_3,i_2',i_3',j_1',j_2',j_3'} A_{j_1,j_2'} B_{i_2',i_3} X_{j_3,j_3'} C_{i_3',i_2} D_{j_2,j_1'} \langle i_1|i_1'\rangle \langle i_2|i_3'\rangle \langle i_3|i_2'\rangle \langle j_1|j_1'\rangle \langle j_2|j_3'\rangle \langle j_3|j_2'\rangle \tag{4.46}$$

$$= \mathrm{Tr}\,(DAX)\,\mathrm{Tr}\,(B)\,\mathrm{Tr}\,(C)\,\langle i_1|i_1'\rangle. \tag{4.47}$$

(23). If $(\pi, \sigma) = ((23), (123))$, then

$$\sum_{i_2,i_3,j_1,j_2,j_3,i_2',i_3',j_1',j_2',j_3'} A_{j_1,j_2'} B_{i_2',i_3} X_{j_3,j_3'} C_{i_3',i_2} D_{j_2,j_1'} \langle i_1|i_1'\rangle \langle i_2|i_3'\rangle \langle i_3|i_2'\rangle \langle j_1|j_2'\rangle \langle j_2|j_3'\rangle \langle j_3|j_1'\rangle \tag{4.48}$$

$$= \mathrm{Tr}\,(A)\,\mathrm{Tr}\,(DX)\,\mathrm{Tr}\,(B)\,\mathrm{Tr}\,(C)\,\langle i_1|i_1'\rangle. \tag{4.49}$$

(24). If $(\pi, \sigma) = ((23), (132))$, then

$$\sum_{i_2,i_3,j_1,j_2,j_3,i_2',i_3',j_1',j_2',j_3'} A_{j_1,j_2'} B_{i_2',i_3} X_{j_3,j_3'} C_{i_3',i_2} D_{j_2,j_1'} \langle i_1|i_1'\rangle \langle i_2|i_3'\rangle \langle i_3|i_2'\rangle \langle j_1|j_3'\rangle \langle j_2|j_1'\rangle \langle j_3|j_2'\rangle \tag{4.50}$$

$$= \mathrm{Tr}\,(D)\,\mathrm{Tr}\,(AX)\,\mathrm{Tr}\,(B)\,\mathrm{Tr}\,(C)\,\langle i_1|i_1'\rangle. \tag{4.51}$$

(25). If $(\pi, \sigma) = ((123), (1))$, then

$$\sum_{i_2,i_3,j_1,j_2,j_3,i_2',i_3',j_1',j_2',j_3'} A_{j_1,j_2'} B_{i_2',i_3} X_{j_3,j_3'} C_{i_3',i_2} D_{j_2,j_1'} \langle i_1|i_2'\rangle \langle i_2|i_3'\rangle \langle i_3|i_1'\rangle \langle j_1|j_1'\rangle \langle j_2|j_2'\rangle \langle j_3|j_3'\rangle \tag{4.52}$$

$$= \mathrm{Tr}\,(AD)\,\mathrm{Tr}\,(X)\,\mathrm{Tr}\,(C)\,\langle i_1\,|B|\,i_1'\rangle. \tag{4.53}$$

(26). If $(\pi, \sigma) = ((123), (12))$, then

$$\sum_{i_2,i_3,j_1,j_2,j_3,i_2',i_3',j_1',j_2',j_3'} A_{j_1,j_2'} B_{i_2',i_3} X_{j_3,j_3'} C_{i_3',i_2} D_{j_2,j_1'} \langle i_1|i_2'\rangle \langle i_2|i_3'\rangle \langle i_3|i_1'\rangle \langle j_1|j_2'\rangle \langle j_2|j_1'\rangle \langle j_3|j_3'\rangle \tag{4.54}$$

$$= \mathrm{Tr}\,(A)\,\mathrm{Tr}\,(D)\,\mathrm{Tr}\,(X)\,\mathrm{Tr}\,(C)\,\langle i_1\,|B|\,i_1'\rangle. \tag{4.55}$$

(27). If $(\pi, \sigma) = ((123), (13))$, then

$$\sum_{i_2,i_3,j_1,j_2,j_3,i_2',i_3',j_1',j_2',j_3'} A_{j_1,j_2'} B_{i_2',i_3} X_{j_3,j_3'} C_{i_3',i_2} D_{j_2,j_1'} \langle i_1|i_2'\rangle \langle i_2|i_3'\rangle \langle i_3|i_1'\rangle \langle j_1|j_3'\rangle \langle j_2|j_2'\rangle \langle j_3|j_1'\rangle \tag{4.56}$$

$$= \mathrm{Tr}\,(ADX)\,\mathrm{Tr}\,(C)\,\langle i_1\,|B|\,i_1'\rangle. \tag{4.57}$$

(28). If $(\pi, \sigma) = ((123), (23))$, then

$$\sum_{i_2,i_3,j_1,j_2,j_3,i_2',i_3',j_1',j_2',j_3'} A_{j_1,j_2'} B_{i_2',i_3} X_{j_3,j_3'} C_{i_3',i_2} D_{j_2,j_1'} \langle i_1|i_2'\rangle \langle i_2|i_3'\rangle \langle i_3|i_1'\rangle \langle j_1|j_1'\rangle \langle j_2|j_3'\rangle \langle j_3|j_2'\rangle \tag{4.58}$$

$$= \mathrm{Tr}\,(DAX)\,\mathrm{Tr}\,(C)\,\langle i_1\,|B|\,i_1'\rangle. \tag{4.59}$$

18

(29). If $(\pi, \sigma) = ((123), (123))$, then

$$\sum_{i_2,i_3,j_1,j_2,j_3,i'_2,i'_3,j'_1,j'_2,j'_3} A_{j_1,j'_2} B_{i'_2,i_3} X_{j_3,j'_3} C_{i'_3,i_2} D_{j_2,j'_1} \langle i_1|i'_2\rangle \langle i_2|i'_3\rangle \langle i_3|i'_1\rangle \langle j_1|j'_2\rangle \langle j_2|j'_3\rangle \langle j_3|j'_1\rangle \tag{4.60}$$

$$= \operatorname{Tr}(A)\operatorname{Tr}(DX)\operatorname{Tr}(C)\langle i_1|B|i'_1\rangle. \tag{4.61}$$

(30). If $(\pi, \sigma) = ((123), (132))$, then

$$\sum_{i_2,i_3,j_1,j_2,j_3,i'_2,i'_3,j'_1,j'_2,j'_3} A_{j_1,j'_2} B_{i'_2,i_3} X_{j_3,j'_3} C_{i'_3,i_2} D_{j_2,j'_1} \langle i_1|i'_2\rangle \langle i_2|i'_3\rangle \langle i_3|i'_1\rangle \langle j_1|j'_3\rangle \langle j_2|j'_1\rangle \langle j_3|j'_2\rangle \tag{4.62}$$

$$= \operatorname{Tr}(D)\operatorname{Tr}(AX)\operatorname{Tr}(C)\langle i_1|B|i'_1\rangle. \tag{4.63}$$

(31). If $(\pi, \sigma) = ((132), (1))$, then

$$\sum_{i_2,i_3,j_1,j_2,j_3,i'_2,i'_3,j'_1,j'_2,j'_3} A_{j_1,j'_2} B_{i'_2,i_3} X_{j_3,j'_3} C_{i'_3,i_2} D_{j_2,j'_1} \langle i_1|i'_3\rangle \langle i_2|i'_1\rangle \langle i_3|i'_2\rangle \langle j_1|j'_1\rangle \langle j_2|j'_2\rangle \langle j_3|j'_3\rangle \tag{4.64}$$

$$= \operatorname{Tr}(AD)\operatorname{Tr}(X)\operatorname{Tr}(B)\langle i_1|C|i'_1\rangle. \tag{4.65}$$

(32). If $(\pi, \sigma) = ((132), (12))$, then

$$\sum_{i_2,i_3,j_1,j_2,j_3,i'_2,i'_3,j'_1,j'_2,j'_3} A_{j_1,j'_2} B_{i'_2,i_3} X_{j_3,j'_3} C_{i'_3,i_2} D_{j_2,j'_1} \langle i_1|i'_3\rangle \langle i_2|i'_1\rangle \langle i_3|i'_2\rangle \langle j_1|j'_2\rangle \langle j_2|j'_1\rangle \langle j_3|j'_3\rangle \tag{4.66}$$

$$= \operatorname{Tr}(A)\operatorname{Tr}(D)\operatorname{Tr}(X)\operatorname{Tr}(B)\langle i_1|C|i'_1\rangle. \tag{4.67}$$

(33). If $(\pi, \sigma) = ((132), (13))$, then

$$\sum_{i_2,i_3,j_1,j_2,j_3,i'_2,i'_3,j'_1,j'_2,j'_3} A_{j_1,j'_2} B_{i'_2,i_3} X_{j_3,j'_3} C_{i'_3,i_2} D_{j_2,j'_1} \langle i_1|i'_3\rangle \langle i_2|i'_1\rangle \langle i_3|i'_2\rangle \langle j_1|j'_3\rangle \langle j_2|j'_2\rangle \langle j_3|j'_1\rangle \tag{4.68}$$

$$= \operatorname{Tr}(ADX)\operatorname{Tr}(B)\langle i_1|C|i'_1\rangle. \tag{4.69}$$

(34). If $(\pi, \sigma) = ((132), (23))$, then

$$\sum_{i_2,i_3,j_1,j_2,j_3,i'_2,i'_3,j'_1,j'_2,j'_3} A_{j_1,j'_2} B_{i'_2,i_3} X_{j_3,j'_3} C_{i'_3,i_2} D_{j_2,j'_1} \langle i_1|i'_3\rangle \langle i_2|i'_1\rangle \langle i_3|i'_2\rangle \langle j_1|j'_1\rangle \langle j_2|j'_3\rangle \langle j_3|j'_2\rangle \tag{4.70}$$

$$= \operatorname{Tr}(DAX)\operatorname{Tr}(B)\langle i_1|C|i'_1\rangle. \tag{4.71}$$

(35). If $(\pi, \sigma) = ((132), (123))$, then

$$\sum_{i_2,i_3,j_1,j_2,j_3,i'_2,i'_3,j'_1,j'_2,j'_3} A_{j_1,j'_2} B_{i'_2,i_3} X_{j_3,j'_3} C_{i'_3,i_2} D_{j_2,j'_1} \langle i_1|i'_3\rangle \langle i_2|i'_1\rangle \langle i_3|i'_2\rangle \langle j_1|j'_2\rangle \langle j_2|j'_3\rangle \langle j_3|j'_1\rangle \tag{4.72}$$

$$= \operatorname{Tr}(A)\operatorname{Tr}(DX)\operatorname{Tr}(B)\langle i_1|C|i'_1\rangle. \tag{4.73}$$

(36). If $(\pi, \sigma) = ((132), (132))$, then

$$\sum_{i_2,i_3,j_1,j_2,j_3,i'_2,i'_3,j'_1,j'_2,j'_3} A_{j_1,j'_2} B_{i'_2,i_3} X_{j_3,j'_3} C_{i'_3,i_2} D_{j_2,j'_1} \langle i_1|i'_3\rangle \langle i_2|i'_1\rangle \langle i_3|i'_2\rangle \langle j_1|j'_3\rangle \langle j_2|j'_1\rangle \langle j_3|j'_2\rangle \tag{4.74}$$

$$= \operatorname{Tr}(D)\operatorname{Tr}(AX)\operatorname{Tr}(B)\langle i_1|C|i'_1\rangle. \tag{4.75}$$

Combing the 36 cases together gives the desired conclusion:

$$\int UAU^\dagger BUXU^\dagger CUDU^\dagger \mathrm{d}\mu(U) = \mu_1 \cdot \mathbb{1}_d + \mu_2 \cdot BC + \mu_3 \cdot CB + \mu_4 \cdot B + \mu_5 \cdot C, \tag{4.76}$$

19

where the coefficients $\mu_j (j = 1, \ldots, 5)$ are given below:

$$
\begin{aligned}
\mu_1 \;:=\; & \mathrm{Wg}(1,1,1)\,\mathrm{Tr}\,(AD)\,\mathrm{Tr}\,(X)\,\mathrm{Tr}\,(BC) + \mathrm{Wg}(2,1)\,\mathrm{Tr}\,(A)\,\mathrm{Tr}\,(D)\,\mathrm{Tr}\,(X)\,\mathrm{Tr}\,(BC) \\
& + \mathrm{Wg}(2,1)\,\mathrm{Tr}\,(ADX)\,\mathrm{Tr}\,(BC) + \mathrm{Wg}(2,1)\,\mathrm{Tr}\,(DAX)\,\mathrm{Tr}\,(BC) \\
& + \mathrm{Wg}(3)\,\mathrm{Tr}\,(A)\,\mathrm{Tr}\,(DX)\,\mathrm{Tr}\,(BC) + \mathrm{Wg}(3)\,\mathrm{Tr}\,(D)\,\mathrm{Tr}\,(AX)\,\mathrm{Tr}\,(BC) \\
& + \mathrm{Wg}(2,1)\,\mathrm{Tr}\,(AD)\,\mathrm{Tr}\,(X)\,\mathrm{Tr}\,(B)\,\mathrm{Tr}\,(C) + \mathrm{Wg}(3)\,\mathrm{Tr}\,(A)\,\mathrm{Tr}\,(D)\,\mathrm{Tr}\,(X)\,\mathrm{Tr}\,(B)\,\mathrm{Tr}\,(C) \\
& + \mathrm{Wg}(3)\,\mathrm{Tr}\,(ADX)\,\mathrm{Tr}\,(B)\,\mathrm{Tr}\,(C) + \mathrm{Wg}(1,1,1)\,\mathrm{Tr}\,(DAX)\,\mathrm{Tr}\,(B)\,\mathrm{Tr}\,(C) \\
& + \mathrm{Wg}(2,1)\,\mathrm{Tr}\,(A)\,\mathrm{Tr}\,(DX)\,\mathrm{Tr}\,(B)\,\mathrm{Tr}\,(C) + \mathrm{Wg}(2,1)\,\mathrm{Tr}\,(D)\,\mathrm{Tr}\,(AX)\,\mathrm{Tr}\,(B)\,\mathrm{Tr}\,(C)\,, && (4.77) \\
\mu_2 \;:=\; & \mathrm{Wg}(2,1)\,\mathrm{Tr}\,(AD)\,\mathrm{Tr}\,(X) + \mathrm{Wg}(1,1,1)\,\mathrm{Tr}\,(A)\,\mathrm{Tr}\,(D)\,\mathrm{Tr}\,(X) + \mathrm{Wg}(3)\,\mathrm{Tr}\,(ADX) \\
& + \mathrm{Wg}(3)\,\mathrm{Tr}\,(DAX) + \mathrm{Wg}(2,1)\,\mathrm{Tr}\,(A)\,\mathrm{Tr}\,(DX) + \mathrm{Wg}(2,1)\,\mathrm{Tr}\,(D)\,\mathrm{Tr}\,(AX)\,, && (4.78) \\
\mu_3 \;:=\; & \mathrm{Wg}(2,1)\,\mathrm{Tr}\,(AD)\,\mathrm{Tr}\,(X) + \mathrm{Wg}(3)\,\mathrm{Tr}\,(A)\,\mathrm{Tr}\,(D)\,\mathrm{Tr}\,(X) + \mathrm{Wg}(1,1,1)\,\mathrm{Tr}\,(ADX) \\
& + \mathrm{Wg}(3)\,\mathrm{Tr}\,(DAX) + \mathrm{Wg}(2,1)\,\mathrm{Tr}\,(A)\,\mathrm{Tr}\,(DX) + \mathrm{Wg}(2,1)\,\mathrm{Tr}\,(D)\,\mathrm{Tr}\,(AX)\,, && (4.79) \\
\mu_4 \;:=\; & \mathrm{Wg}(3)\,\mathrm{Tr}\,(AD)\,\mathrm{Tr}\,(X)\,\mathrm{Tr}\,(C) + \mathrm{Wg}(2,1)\,\mathrm{Tr}\,(A)\,\mathrm{Tr}\,(D)\,\mathrm{Tr}\,(X)\,\mathrm{Tr}\,(C) \\
& + \mathrm{Wg}(2,1)\,\mathrm{Tr}\,(ADX)\,\mathrm{Tr}\,(C) + \mathrm{Wg}(2,1)\,\mathrm{Tr}\,(DAX)\,\mathrm{Tr}\,(C) \\
& + \mathrm{Wg}(1,1,1)\,\mathrm{Tr}\,(A)\,\mathrm{Tr}\,(DX)\,\mathrm{Tr}\,(C) + \mathrm{Wg}(3)\,\mathrm{Tr}\,(D)\,\mathrm{Tr}\,(AX)\,\mathrm{Tr}\,(C)\,, && (4.80) \\
\mu_5 \;:=\; & \mathrm{Wg}(3)\,\mathrm{Tr}\,(AD)\,\mathrm{Tr}\,(X)\,\mathrm{Tr}\,(B) + \mathrm{Wg}(2,1)\,\mathrm{Tr}\,(A)\,\mathrm{Tr}\,(D)\,\mathrm{Tr}\,(X)\,\mathrm{Tr}\,(B) \\
& + \mathrm{Wg}(2,1)\,\mathrm{Tr}\,(ADX)\,\mathrm{Tr}\,(B) + \mathrm{Wg}(2,1)\,\mathrm{Tr}\,(DAX)\,\mathrm{Tr}\,(B) \\
& + \mathrm{Wg}(3)\,\mathrm{Tr}\,(A)\,\mathrm{Tr}\,(DX)\,\mathrm{Tr}\,(B) + \mathrm{Wg}(1,1,1)\,\mathrm{Tr}\,(D)\,\mathrm{Tr}\,(AX)\,\mathrm{Tr}\,(B)\,. && (4.81)
\end{aligned}
$$

We are done. $\qquad\qquad\square$

Remark 4.1. $\mathrm{Wg} := \frac{1}{(k!)^2} \sum_{\lambda \vdash k} \frac{\dim(\mathbf{P}_\lambda)^2}{\dim(\mathbf{Q}_\lambda)} \chi_\lambda$ is called the *Weingarten function* [4]. In particular, for $\lambda \vdash 3$, we have:

$$
\begin{aligned}
\mathrm{Wg}(1,1,1) &= \frac{d^2 - 2}{d(d^2-1)(d^2-4)} = \left(d - \frac{2}{d}\right) \cdot \frac{1}{N_d}, && (4.82) \\
\mathrm{Wg}(2,1) &= -\frac{1}{(d^2-1)(d^2-4)} = (-1) \cdot \frac{1}{N_d}, && (4.83) \\
\mathrm{Wg}(3) &= \frac{2}{d(d^2-1)(d^2-4)} = \frac{2}{d} \cdot \frac{1}{N_d}, && (4.84)
\end{aligned}
$$

where $N_d = (d^2 - 1)(d^2 - 4)$. With these coefficients, we then have

$$
\begin{aligned}
N_d \mu_1 \; := \; & \left(d - \frac{2}{d}\right) \operatorname{Tr}(AD)\operatorname{Tr}(X)\operatorname{Tr}(BC) + \left(d - \frac{2}{d}\right)\operatorname{Tr}(DAX)\operatorname{Tr}(B)\operatorname{Tr}(C) + \frac{2}{d}\operatorname{Tr}(A)\operatorname{Tr}(DX)\operatorname{Tr}(BC) \\
& + \frac{2}{d}\operatorname{Tr}(D)\operatorname{Tr}(AX)\operatorname{Tr}(BC) + \frac{2}{d}\operatorname{Tr}(A)\operatorname{Tr}(D)\operatorname{Tr}(X)\operatorname{Tr}(B)\operatorname{Tr}(C) + \frac{2}{d}\operatorname{Tr}(ADX)\operatorname{Tr}(B)\operatorname{Tr}(C) \\
& - \operatorname{Tr}(A)\operatorname{Tr}(D)\operatorname{Tr}(X)\operatorname{Tr}(BC) - \operatorname{Tr}(ADX)\operatorname{Tr}(BC) - \operatorname{Tr}(DAX)\operatorname{Tr}(BC) \\
& - \operatorname{Tr}(AD)\operatorname{Tr}(X)\operatorname{Tr}(B)\operatorname{Tr}(C) - \operatorname{Tr}(A)\operatorname{Tr}(DX)\operatorname{Tr}(B)\operatorname{Tr}(C) - \operatorname{Tr}(D)\operatorname{Tr}(AX)\operatorname{Tr}(B)\operatorname{Tr}(C) \quad (4.85)
\end{aligned}
$$

$$
\begin{aligned}
N_d \mu_2 \; := \; & \left(d - \frac{2}{d}\right)\operatorname{Tr}(A)\operatorname{Tr}(D)\operatorname{Tr}(X) + \frac{2}{d}\operatorname{Tr}(ADX) + \frac{2}{d}\operatorname{Tr}(DAX) \\
& - \operatorname{Tr}(AD)\operatorname{Tr}(X) - \operatorname{Tr}(A)\operatorname{Tr}(DX) - \operatorname{Tr}(D)\operatorname{Tr}(AX), \quad (4.86)
\end{aligned}
$$

$$
\begin{aligned}
N_d \mu_3 \; := \; & \frac{2}{d}\operatorname{Tr}(DAX) + \frac{2}{d}\operatorname{Tr}(A)\operatorname{Tr}(D)\operatorname{Tr}(X) + \left(d - \frac{2}{d}\right)\operatorname{Tr}(ADX) \\
& - \operatorname{Tr}(AD)\operatorname{Tr}(X) - \operatorname{Tr}(A)\operatorname{Tr}(DX) - \operatorname{Tr}(D)\operatorname{Tr}(AX), \quad (4.87)
\end{aligned}
$$

$$
\begin{aligned}
N_d \mu_4 \; := \; & \left(d - \frac{2}{d}\right)\operatorname{Tr}(A)\operatorname{Tr}(DX)\operatorname{Tr}(C) + \frac{2}{d}\operatorname{Tr}(AD)\operatorname{Tr}(X)\operatorname{Tr}(C) + \frac{2}{d}\operatorname{Tr}(D)\operatorname{Tr}(AX)\operatorname{Tr}(C) \\
& - \operatorname{Tr}(A)\operatorname{Tr}(D)\operatorname{Tr}(X)\operatorname{Tr}(C) - \operatorname{Tr}(ADX)\operatorname{Tr}(C) - \operatorname{Tr}(DAX)\operatorname{Tr}(C), \quad (4.88)
\end{aligned}
$$

$$
\begin{aligned}
N_d \mu_5 \; := \; & \frac{2}{d}\operatorname{Tr}(AD)\operatorname{Tr}(X)\operatorname{Tr}(B) + \frac{2}{d}\operatorname{Tr}(A)\operatorname{Tr}(DX)\operatorname{Tr}(B) + \left(d - \frac{2}{d}\right)\operatorname{Tr}(D)\operatorname{Tr}(AX)\operatorname{Tr}(B) \\
& - \operatorname{Tr}(A)\operatorname{Tr}(D)\operatorname{Tr}(X)\operatorname{Tr}(B) - \operatorname{Tr}(ADX)\operatorname{Tr}(B) - \operatorname{Tr}(DAX)\operatorname{Tr}(B). \quad (4.89)
\end{aligned}
$$

B. The proof of Theorem 2.2

By Lemma 2.1,

$$
\int U^\dagger M_{ij} U T U^\dagger M_{jl} U T U^\dagger M_{li} U \mathrm{d}\mu(U) = f(i,j,l) \cdot \mathbb{1}_d + g(i,j,l) \cdot T + h(i,j,l) \cdot T^2,
$$

where

$$
\begin{aligned}
f(i,j,l) \; = \; & \mathrm{Wg}(1,1,1)d_A^2 \delta_{jl} \operatorname{Tr}\left(T^2\right) + \mathrm{Wg}(2,1)d_A^3 \delta_{ij}\delta_{jl}\delta_{li} \operatorname{Tr}\left(T^2\right) + \mathrm{Wg}(2,1)d_A \delta_{ij}\delta_{jl}\delta_{li} \operatorname{Tr}\left(T^2\right) \\
& + \mathrm{Wg}(2,1)d_A \operatorname{Tr}\left(T^2\right) + \mathrm{Wg}(3)d_A^2 \delta_{ij} \operatorname{Tr}\left(T^2\right) + \mathrm{Wg}(3)d_A^2 \delta_{il} \operatorname{Tr}\left(T^2\right) \\
& + \mathrm{Wg}(2,1)d_A^2 \delta_{jl} \operatorname{Tr}(T)^2 + \mathrm{Wg}(3)d_A^3 \delta_{ij}\delta_{jl}\delta_{li} \operatorname{Tr}(T)^2 + \mathrm{Wg}(3)d_A \delta_{ij}\delta_{jl}\delta_{li} \operatorname{Tr}(T)^2 \\
& + \mathrm{Wg}(1,1,1)d_A \operatorname{Tr}(T)^2 + \mathrm{Wg}(2,1)d_A^2 \delta_{ij} \operatorname{Tr}(T)^2 + \mathrm{Wg}(2,1)d_A^2 \delta_{il} \operatorname{Tr}(T)^2, \\
g(i,j,l) \; = \; & \mathrm{Wg}(3)d_A^2 \delta_{jl} \operatorname{Tr}(T) + \mathrm{Wg}(2,1)d_A^3 \delta_{ij}\delta_{jl}\delta_{li} \operatorname{Tr}(T) + \mathrm{Wg}(2,1)d_A \delta_{ij}\delta_{jl}\delta_{li} \operatorname{Tr}(T) \\
& + \mathrm{Wg}(2,1)d_A \operatorname{Tr}(T) + \mathrm{Wg}(1,1,1)d_A^2 \delta_{ij} \operatorname{Tr}(T) + \mathrm{Wg}(3)d_A^2 \delta_{il} \operatorname{Tr}(T) \\
& + \mathrm{Wg}(3)d_A^2 \delta_{jl} \operatorname{Tr}(T) + \mathrm{Wg}(2,1)d_A^3 \delta_{ij}\delta_{jl}\delta_{li} \operatorname{Tr}(T) + \mathrm{Wg}(2,1)d_A \delta_{ij}\delta_{jl}\delta_{li} \operatorname{Tr}(T) \\
& + \mathrm{Wg}(2,1)d_A \operatorname{Tr}(T) + \mathrm{Wg}(3)d_A^2 \delta_{ij} \operatorname{Tr}(T) + \mathrm{Wg}(1,1,1)d_A^2 \delta_{il} \operatorname{Tr}(T), \\
h(i,j,l) \; = \; & \mathrm{Wg}(2,1)d_A^2 \delta_{jl} + \mathrm{Wg}(1,1,1)d_A^3 \delta_{ij}\delta_{jl}\delta_{li} + \mathrm{Wg}(3)d_A \delta_{ij}\delta_{jl}\delta_{li} \\
& + \mathrm{Wg}(3)d_A + \mathrm{Wg}(2,1)d_A^2 \delta_{ij} + \mathrm{Wg}(2,1)d_A^2 \delta_{il}, \\
& + \mathrm{Wg}(2,1)d_A^2 \delta_{jl} + \mathrm{Wg}(3)d_A^3 \delta_{ij}\delta_{jl}\delta_{li} + \mathrm{Wg}(1,1,1)d_A \delta_{ij}\delta_{jl}\delta_{li} \\
& + \mathrm{Wg}(3)d_A + \mathrm{Wg}(2,1)d_A^2 \delta_{ij} + \mathrm{Wg}(2,1)d_A^2 \delta_{il}.
\end{aligned}
$$

Note that the meaning of the notation $\text{Wg}(*)$ can be found in the Appendix. Thus for $f := \sum_{i,j,l=1}^{d_B} f(i,j,l)$, $g := \sum_{i,j,l=1}^{d_B} g(i,j,l)$, and $h := \sum_{i,j,l=1}^{d_B} h(i,j,l)$, we have

$$
\begin{aligned}
f &= \Big([\text{Wg}(1,1,1) + 2\text{Wg}(3)]d^2 + \text{Wg}(2,1)(dd_A^2 + d + dd_B^2)\Big)\,\text{Tr}\left(T^2\right)\\
&\quad + \Big(3\text{Wg}(2,1)d^2 + \text{Wg}(3)(dd_A^2 + d) + \text{Wg}(1,1,1)dd_B^2\Big)\,\text{Tr}\left(T\right)^2,\\
g &= \Big([4\text{Wg}(3) + 2\text{Wg}(1,1,1)]d^2 + 2\text{Wg}(2,1)(dd_A^2 + d + dd_B^2)\Big)\,\text{Tr}\left(T\right),\\
h &= 6\text{Wg}(2,1)d^2 + [\text{Wg}(1,1,1) + \text{Wg}(3)]dd_A^2 + [\text{Wg}(1,1,1) + \text{Wg}(3)]d + 2\text{Wg}(3)dd_B^2.
\end{aligned}
$$

Hence, for $T = \mathbb{1}_A \otimes \mathbb{1}_B / d_B - \rho_{AB}$, $\text{Tr}\left(T\right) = d_A - 1$ and $\text{Tr}\left(T^2\right) = \frac{d_A - 2}{d_B} + \text{Tr}\left(\rho_{AB}^2\right)$, then

$$
\begin{aligned}
f &= \frac{d(d^2 - d_A^2 - d_B^2 + 1)}{(d^2 - 1)(d^2 - 4)}\,\text{Tr}\left(T^2\right) + \frac{d^2(d_B^2 - 3) + 2(d_A^2 - d_B^2 + 1)}{(d^2 - 1)(d^2 - 4)}\,\text{Tr}\left(T\right)^2, & (4.90)\\
&= \frac{d_A(d_A^2 - 1)(d_B^2 - 1)}{(d^2 - 1)(d^2 - 4)}\left(d_A + d_B\,\text{Tr}\left(\rho_{AB}^2\right) - 2\right) & (4.91)\\
&\quad + \frac{(d^2 - 2d_A^2 - 2)(d_B^2 - 1)}{(d^2 - 1)(d^2 - 4)}(d_A - 1)^2, & (4.92)\\
g &= \frac{2d(d^2 - d_A^2 - d_B^2 + 1)}{(d^2 - 1)(d^2 - 4)}\,\text{Tr}\left(T\right) = \frac{2d(d_A - 1)(d_A^2 - 1)(d_B^2 - 1)}{(d^2 - 1)(d^2 - 4)}, & (4.93)\\
h &= \frac{d^2(d_A^2 - 5) + 4d_B^2}{(d^2 - 1)(d^2 - 4)} = \frac{(d_A^2 - 1)(d_A^2 - 4)d_B^2}{(d^2 - 1)(d^2 - 4)}. & (4.94)
\end{aligned}
$$

Therefore,

$$
\int U^\dagger \left[\Gamma(UTU^\dagger)\right]^2 U \mathrm{d}\mu(U) = f \cdot \mathbb{1}_d + g \cdot T + h \cdot T^2, \tag{4.95}
$$

implying that

$$
\begin{aligned}
a_2 &= f + g \cdot \text{Tr}\left(\rho_{AB}T\right) + h \cdot \text{Tr}\left(\rho_{AB}T^2\right)\\
&= f + g \cdot \left(\frac{1}{d_B} - \text{Tr}\left(\rho_{AB}^2\right)\right) + h \cdot \left(\frac{1}{d_B^2} - \frac{2}{d_B}\text{Tr}\left(\rho_{AB}^2\right) + \text{Tr}\left(\rho_{AB}^3\right)\right)\\
&= \left(f + g\frac{1}{d_B} + h\frac{1}{d_B^2}\right) - \left(g + h\frac{2}{d_B}\right)\text{Tr}\left(\rho_{AB}^2\right) + h\,\text{Tr}\left(\rho_{AB}^3\right). \tag{4.96}
\end{aligned}
$$

We make further analysis of the term a_n although we have already known the fact that $\lim_{n\to\infty} a_n = 0$. Let $\varphi_\rho(X) = \text{Tr}\left(\rho X\right)$. Apparently, φ_ρ is a positive unital linear mapping (in fact, it is a positive unital linear functional from the set of $n \times n$ Hermitian matrices to $\mathbb{R}$). It is easily seen that $f(x) = x^n$ is a convex function from $\mathbb{R}$ to $\mathbb{R}$. By using [10, Theorem 4.15, pp147], we see that

$$
\begin{aligned}
a_n &= \varphi_\rho\left(\int \left[U^\dagger\Gamma(UTU^\dagger)U\right]^n \mathrm{d}\mu(U)\right) = \varphi_\rho\left(\int f(U^\dagger\Gamma(UTU^\dagger)U)\mathrm{d}\mu(U)\right)\\
&= \int \mathrm{d}\mu(U)(\varphi_\rho \circ f)(U^\dagger\Gamma(UTU^\dagger)U) \geqslant \int \mathrm{d}\mu(U)f[\varphi_\rho(U^\dagger\Gamma(UTU^\dagger)U)]\\
&\geqslant \int \mathrm{d}\mu(U)\left[\text{Tr}\left(\rho U^\dagger\Gamma(UTU^\dagger)U\right)\right]^n \geqslant \left(\text{Tr}\left(\rho\int \mathrm{d}\mu(U)U^\dagger\Gamma(UTU^\dagger)U\right)\right)^n.
\end{aligned}
$$

By (2.4), we have

$$\text{Tr}\left(\rho \int d\mu(U) U^\dagger \Gamma(UTU^\dagger) U\right) = \frac{d_A - 1}{d^2 - 1}\left[(1 + dd_B) - (d + d_B)\,\text{Tr}\left(\rho_{AB}^2\right)\right] = a_1.$$

Thus

$$a_n \;\geqslant\; a_1^n. \tag{4.97}$$

Then

$$a_1 = \frac{(d_A - 1)(d_B - 1)}{d + 1} + \frac{(d_A - 1)(d + d_B)}{d^2 - 1}S_L(\rho_{AB}),$$

where $S_L(\rho_{AB}) \in \left[0, 1 - \frac{1}{d}\right]$. Clearly

$$\frac{(d_A - 1)(d_B - 1)}{d + 1} \leqslant a_1 \leqslant \frac{(d_A - 1)\left(1 + \frac{1}{d_A}\right)}{d + 1} < 1. \tag{4.98}$$

Now

$$\int S(\rho_A') d\mu(U) = \sum_{n=1}^{\infty} \frac{a_n}{n} \geqslant \sum_{n=1}^{\infty} \frac{a_1^n}{n} = -\ln(1 - a_1). \tag{4.99}$$

We see from the above lower bound, i.e., $-\ln(1 - a_1)$, that when the purity of ρ_{AB} decreases, a_1 increases. Hence such lower bound will be tighter.

C. The proof of Proposition 3.2

Clearly, the first inequality is easily obtained

$$\int I(A : B)_{\rho'} d\mu(U) \;\geqslant\; S\left(\rho_{AB} \big\| \int \ln\left[U^\dagger(\rho_A' \otimes \rho_B')U\right] d\mu(U)\right) \tag{4.100}$$

$$= S\left(\rho_{AB} \big\| c_0 \cdot \mathbb{1}_d + c_1 \cdot \rho_{AB} + c_2 \cdot \rho_{AB}^2\right). \tag{4.101}$$

In order to show the second inequality, note that, for any two density matrices ρ and σ,

$$F(\rho, \sigma) \geqslant \text{Tr}\left(\sqrt{\rho}\sqrt{\sigma}\right) \geqslant \text{Tr}\left(\rho\sigma\right). \tag{4.102}$$

Then, for $\rho_{AB}' = U\rho_{AB}U^\dagger$, we have

$$F(\rho_{AB}', \rho_A' \otimes \rho_B') \geqslant \text{Tr}\left(\rho_{AB}'\rho_A' \otimes \rho_B'\right) = \text{Tr}\left(\rho_{AB}U^\dagger\left(\rho_A' \otimes \rho_B'\right)U\right), \tag{4.103}$$

implying that

$$\int F(\rho_{AB}', \rho_A' \otimes \rho_B') d\mu(U) \geqslant \int \text{Tr}\left(\rho_{AB}U^\dagger\left(\rho_A' \otimes \rho_B'\right)U\right) d\mu(U) = c_0 + c_1\,\text{Tr}\left(\rho_{AB}^2\right) + c_2\,\text{Tr}\left(\rho_{AB}^3\right). \tag{4.104}$$

By the concavity of fidelity, we have

$$\int F(\rho_{AB}', \rho_A' \otimes \rho_B') d\mu(U) = \int F(\rho_{AB}, U^\dagger(\rho_A' \otimes \rho_B')U) d\mu(U) \leqslant F(\rho_{AB}, c_0 \cdot \mathbb{1}_d + c_1 \cdot \rho_{AB} + c_2 \cdot \rho_{AB}^2). \tag{4.105}$$

Therefore

$$c_0 + c_1\,\text{Tr}\left(\rho_{AB}^2\right) + c_2\,\text{Tr}\left(\rho_{AB}^3\right) \leqslant \int F(\rho_{AB}', \rho_A' \otimes \rho_B') d\mu(U) \leqslant F(\rho_{AB}, c_0 \cdot \mathbb{1}_d + c_1 \cdot \rho_{AB} + c_2 \cdot \rho_{AB}^2). \tag{4.106}$$

This completes the proof.

D. The proof of Theorem 3.4

Note that $\rho'_{AB} = U\rho_{AB}U^\dagger$. We see from (3.8) that

$$\int I(A:B)_{\rho'}\mathrm{d}\mu(U) \geqslant S\left(\rho_{AB}||c_0 \cdot \mathbb{1}_d + c_1 \cdot \rho_{AB} + c_2 \cdot \rho_{AB}^2\right). \tag{4.107}$$

Since $I(A:B)_{\rho'} = S(\rho'_A) + S(\rho'_B) - S(\rho_{AB})$, it follows that

$$\int \left(S(\rho'_A) + S(\rho'_B) - S(\rho_{AB})\right)\mathrm{d}\mu(U) \geqslant S\left(\rho_{AB}||c_0 \cdot \mathbb{1}_d + c_1 \cdot \rho_{AB} + c_2 \cdot \rho_{AB}^2\right). \tag{4.108}$$

That is,

$$\langle S_A + S_B \rangle \geqslant S(\rho_{AB}) + S\left(\rho_{AB}||c_0 \cdot \mathbb{1}_d + c_1 \cdot \rho_{AB} + c_2 \cdot \rho_{AB}^2\right). \tag{4.109}$$

This confirms the first inequality. Besides, by Eq. (3.4), we get

$$S\left(c_0 \cdot \mathbb{1}_d + c_1 \cdot \rho_{AB} + c_2 \cdot \rho_{AB}^2\right) = S\left(\int U^\dagger(\rho'_A \otimes \rho'_B)U\mathrm{d}\mu(U)\right) \tag{4.110}$$

$$\geqslant \int S\left(U^\dagger(\rho'_A \otimes \rho'_B)U\right)\mathrm{d}\mu(U) = \int S(\rho'_A)\mathrm{d}\mu(U) + \int S(\rho'_B)\mathrm{d}\mu(U). \tag{4.111}$$

This confirms the second inequality. Therefore we have

$$S(\rho_{AB}) + S\left(\rho_{AB}||c_0 \cdot \mathbb{1}_d + c_1 \cdot \rho_{AB} + c_2 \cdot \rho_{AB}^2\right) \leqslant S\left(c_0 \cdot \mathbb{1}_d + c_1 \cdot \rho_{AB} + c_2 \cdot \rho_{AB}^2\right).$$

This is equivalent to the following

$$S\left(\rho_{AB}||c_0 \cdot \mathbb{1}_d + c_1 \cdot \rho_{AB} + c_2 \cdot \rho_{AB}^2\right) \leqslant S\left(c_0 \cdot \mathbb{1}_d + c_1 \cdot \rho_{AB} + c_2 \cdot \rho_{AB}^2\right) - S(\rho_{AB}).$$

Next we show that $S\left(c_0 \cdot \mathbb{1}_d + c_1 \cdot \rho_{AB} + c_2 \cdot \rho_{AB}^2\right) = S(\rho_{AB})$ if and only if ρ_{AB} is maximally mixed state. Clearly if ρ_{AB} is maximally mixed state, i.e., $S(\rho_{AB}) = \ln(d)$, since $S\left(c_0 \cdot \mathbb{1}_d + c_1 \cdot \rho_{AB} + c_2 \cdot \rho_{AB}^2\right) - S(\rho_{AB}) \geqslant 0$, then $S\left(c_0 \cdot \mathbb{1}_d + c_1 \cdot \rho_{AB} + c_2 \cdot \rho_{AB}^2\right) \geqslant \ln(d)$, apparently $S\left(c_0 \cdot \mathbb{1}_d + c_1 \cdot \rho_{AB} + c_2 \cdot \rho_{AB}^2\right) \leqslant \ln(d)$, thus $S(\rho_{AB}) = S\left(c_0 \cdot \mathbb{1}_d + c_1 \cdot \rho_{AB} + c_2 \cdot \rho_{AB}^2\right) = \ln(d)$, the maximum of von Neuman entropy. Reversely, if $S\left(c_0 \cdot \mathbb{1}_d + c_1 \cdot \rho_{AB} + c_2 \cdot \rho_{AB}^2\right) = S(\rho_{AB})$, then by the obtained inequality, we have

$$S\left(\rho_{AB}||c_0 \cdot \mathbb{1}_d + c_1 \cdot \rho_{AB} + c_2 \cdot \rho_{AB}^2\right) = 0,$$

which holds if and only if $\rho_{AB} = c_0 \cdot \mathbb{1}_d + c_1 \cdot \rho_{AB} + c_2 \cdot \rho_{AB}^2$. This means that for any eigenvalue $\lambda(\geqslant 0)$ of ρ_{AB} must satisfy that

$$c_2\lambda^2 + (c_1 - 1)\lambda + c_0 = 0.$$

Solve this equation, we get

$$\lambda = \frac{(1-c_1) - \sqrt{(1-c_1)^2 - 4c_0c_2}}{2c_2} = \frac{1}{d}.$$

Note that we have dropped another root being larger than one. Thus ρ_{AB} is maximally mixed state. In fact, we get that $\rho_{AB} = c_0 \cdot \mathbb{1}_d + c_1 \cdot \rho_{AB} + c_2 \cdot \rho_{AB}^2$ if and only if ρ_{AB} is maximally mixed state.

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